THÈSE EN COTUTELLE
ENS CACHAN - BRETAGNE / CHARLES UNIVERSITY IN PRAGUE
Pour obtenir le diplôme de
DOCTEUR DE L'ÉCOLE NORMALE SUPÉRIEURE DE CACHAN
Mention : Mathématiques
sous le sceau de l'Université européenne de Bretagne
École doctorale MATISSE

présentée par
Martina Hofmanova
Préparée à l'Institut de Recherche Mathématiques de Rennes et au département d'analyse mathématique, Faculty of Mathematics and Physics, Charles University in Prague

Degenerate Parabolic Stochastic Partial Differential Equations

Thèse soutenue le 5 juillet 2013
devant le jury composé de :

Franco Flandoli
Professeur, Università di Pisa / *rapporteur*

Benoît Perthame
Professeur, Université Pierre et Marie Curie / *rapporteur*

Bohdan Maslowski
Professeur, Charles University in Prague / *examinateur*

Florian Méhats
Professeur, Université de Rennes 1 / *examinateur*

Julien Vovelle
Chargé de recherches, Université Claude Bernard Lyon 1 / *examinateur*

Arnaud Debussche
Professeur, ENS Cachan-Bretagne / *directeur de thèse*

Jan Seidler
Directeur de recherches, Institute of Information Theory and Automation AS CR / *directeur de thèse*

Charles University in Prague, Faculty of Mathematics and Physics
Department of Mathematical Analysis, Czech Republic
École Normale Supérieure de Cachan, Antenne de Bretagne, France

DOCTORAL THESIS

Martina Hofmanová

Degenerate Parabolic Stochastic Partial Differential Equations

Supervisors of the doctoral thesis:
Prof. Arnaud Debussche, École Normale Supérieure de Cachan, Antenne de Bretagne, France
RNDr. Jan Seidler, CSc., Institute of Information Theory and Automation, Academy of Science Czech Republic

Study programme: Mathematics
Specialization: Mathematical Analysis

2013

Acknowledgments

First and foremost, I would like to express my deepest gratitude to Arnaud Debussche for being an outstanding advisor: Thank you for making me understand that research is fun and for staying positive all the time no matter what, it always gave me the motivation I needed.

My sincere thanks go also to Jan Seidler for bringing me to the idea of joint PhD between Czech Republic and France as well as for helping me through the whole process.

I am particularly grateful to Julien Vovelle for his interest in my work as well as the number of remarks and suggestions he provided; to Bohdan Maslowski, Josef Málek and Šárka Nečasová for their support. I am greatly indebted to Franco Flandoli and Benoit Perthame for their time and effort in reviewing this work and to Florian Méhats for being part of the jury.

I recognize that this research would not have been possible without the financial assistance of several institutions, I acknowledge: the French Government; École Normale Supérieure de Cachan, Antenne de Bretagne; Charles University in Prague; Institute of Information Theory and Automation AS CR; University Center for Mathematical Modelling, Applied Analysis and Computational Mathematics; Inria, Centre de Recherche, Rennes – Bretagne Atlantique and the Women in Science fellowship "La bourse L'Oréal France – Unesco – Académie des Science – Pour les femmes et la science".

Last but not least, I would like to thank my family, all my friends and colleagues, and especially to Adam and Archie: I'm very happy that you decided to keep me...

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.
I understand that my work relates to the rights and obligations under the Act No. 121/2000 Coll., the Copyright Act, as amended, in particular the fact that the Charles University in Prague has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 paragraph 1 of the Copyright Act.

In Rennes, Martina Hofmanová

Název práce: Degenerované parabolické stochastické parciální diferenciální rovnice
Autor: Martina Hofmanová
Ústav: Ústav Teorie Informace a Automatizace, v.v.i., Akademie věd České Republiky
Vedoucí doktorské práce:

- Prof. Arnaud Debussche, École Normale Supérieure de Cachan, Antenne de Bretagne, France
- RNDr. Jan Seidler, CSc., Ústav Teorie Informace a Automatizace, v.v.i., Akademie věd České Republiky

Abstrakt: Tato disertace se zaměřuje na několik problémů, které vyvstávají při studiu degenerovaných parabolických stochastických parcialních diferenciálních rovnic, stochastických hyperbolických zákonů zachování a stochastických diferenciálních rovnic se spojitými koeficienty. V první části studujeme degenerované parabolické stochastické parciální diferenciální rovnice, adaptujeme pojem kinetické formulace a kinetického řešení a ukážeme existenci, jednoznačnost a spojitou závislost na počáteční podmínce. Jako přípravný výsledek pak dokážeme regularitu řešení v nedegenerovaném případě za předpokladu hladkých koeficientů s omezenými derivacemi. Ve druhé části uvažujeme stochastické hyperbolické zákony zachování a studujeme jejich aproximaci ve smyslu Bhatnagar-Gross-Krooka. Konkrétně, popíšeme zákony zachování jakožto hydrodynamickou limitu stochastického BGK modelu jestliže mikroskopická škála jde k nule. V poslední části předkládáme nový a elementární důkaz Skorokhodova klasického výsledku o existenci slabého řešení stochastických diferenciálních rovnic se spojitými koeficienty, jež splňují vhodnou Lyapunovskou podmínku.
Klíčová slova: degenerované parabolické stochastické parciální diferenciální rovnice, kinetické řešení, BGK model, stochastické diferenciální rovnice

Title: Degenerate parabolic stochastic partial differential equations
Author: Martina Hofmanová
Institute: Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic
Supervisors of the doctoral thesis:

- Prof. Arnaud Debussche, École Normale Supérieure de Cachan, Antenne de Bretagne, France
- RNDr. Jan Seidler, CSc., Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic

Abstract: In this thesis, we address several problems arising in the study of nondegenerate and degenerate parabolic SPDEs, stochastic hyperbolic conservation laws and SDEs with continues coefficients. In the first part, we are interested in degenerate parabolic SPDEs, adapt the notion of kinetic formulation and kinetic solution and establish existence, uniqueness as well as continuous dependence on initial data. As a preliminary result we obtain regularity of solutions in the nondegenerate case under the hypothesis that all the coefficients are sufficiently smooth and have bounded derivatives. In the second part, we consider hyperbolic conservation laws with stochastic forcing and study their approximations in the sense of Bhatnagar-Gross-Krook. In particular, we describe the conservation laws as a hydrodynamic limit of the stochastic BGK model as the microscopic scale vanishes. In the last part, we provide a new and fairly elementary proof of Skorkohod's classical theorem on existence of weak solutions to SDEs with continuous coefficients satisfying a suitable Lyapunov condition.
Keywords: degenerate parabolic stochastic partial differential equations, kinetic solution, BGK model, stochastic differential equations

Résumé

Dans cette thèse, on s'intéresse à plusieurs problèmes intervenant dans l'étude d'Équations aux Dérivées Partielles Stochastiques paraboliques, non-dégénérées et dégénérées, de lois de conservation hyperboliques stochastiques, et d'Équations Différentielles Stochastiques avec des coefficients continus.

Dans la première partie, on s'intéresse à des EDPS paraboliques de la forme

$$\begin{aligned} \mathrm{d}u+\operatorname{div}\big(B(u)\big)\,\mathrm{d}t&=\operatorname{div}\big(A(x)\nabla u\big)\mathrm{d}t+\Phi(u)\,\mathrm{d}W, \qquad x\in\mathbb{T}^N,\ t\in(0,T),\\ u(0)&=u_0, \end{aligned} \tag{0.1}$$

où W est un processus de Wiener cylindrique. Sans l'hypothèse que la matrice de diffusion A est définie positive, cette équation peut être dégénérée, ce qui constitue la principale difficulté dans la résolution du problème. On suppose que la matrice A est semi-définie positive, et par conséquent elle peut être identiquement nulle, donnant ainsi une loi de conservation hyperbolique. On adapte les notions de formulation et de solution cinétiques, qui ont été précédemment étudiées dans le cas de lois de conservation hyperboliques scalaires, à la fois dans le contexte déterministe (voir par exemple Imbert et Vovelle [38], Lions, Perthame et Tadmor [55], [56], Perthame [63], [64]) et la situation stochastique par Debussche et Vovelle [16], ainsi que dans le cas d'equations paraboliques dégénérées déterministes du second-ordre par Chen et Perthame [13]. Le concept de solution cinétique s'applique à des situations plus générales que celle de solution entropique introduite par Kružkov [45], et il paraît plus adapté en particulier pour des problèmes paraboliques dégénérés : il permet de conserver la structure précise de la mesure de dissipation parabolique, tandis qu'en utilisant des solutions entropiques une partie de l'information est perdue et doit être retrouvée à un certain moment.

Supposons que u est une solution régulière de (0.1), plus précisément

$$u\in C([0,T];C^2(\mathbb{T}^N))\quad \mathbb{P}\text{-p.s.}.$$

D'après la formule d'Itô, il vient alors que $f(x,t,\xi)=\mathbf{1}_{u(x,t)>\xi}$ vérifie au sens des distributions dans $\mathcal{D}'(\mathbb{T}^N_x\times\mathbb{R}_\xi)$

$$\mathrm{d}f+b\cdot\nabla f\,\mathrm{d}t-\operatorname{div}\big(A\nabla f\big)\,\mathrm{d}t=\delta_{u=\xi}\Phi\,\mathrm{d}W+\partial_\xi\bigg(n_1-\frac{1}{2}G^2\delta_{u=\xi}\bigg)\,\mathrm{d}t,$$

où n_1 est la mesure de dissipation parabolique définie par

$$\mathrm{d}n_1(x,t,\xi)=(\nabla u)^*A(\nabla u)\,\mathrm{d}\delta_{u(x,t)}(\xi)\,\mathrm{d}x\,\mathrm{d}t.$$

Ce problème est généralisé, de telle sorte qu'on obtient la formulation cinétique de (0.1), qui est également faible en temps, et s'écrit formellement

$$\partial_t f+b\cdot\nabla f-\operatorname{div}\big(A\nabla f\big)=\delta_{u=\xi}\,\Phi\,\dot W+\partial_\xi\bigg(m-\frac{1}{2}G^2\delta_{u=\xi}\bigg). \tag{0.2}$$

On recherche un couple (f,m), m étant une mesure cinétique - i.e. une mesure de Borel positive bornée aléatoire sur $\mathbb{T}^N\times[0,T]\times\mathbb{R}$ - obtenue comme somme de deux composantes n_1+n_2 : n_1 est la mesure de dissipation parabolique mentionnée précédemment, et n_2 est une mesure inconnue, de façon à prendre en compte de possibles singularités, et s'annulant dans le cas non-dégénéré. Une solution cinétique est alors définie comme

suit : soit $u \in L^p(\Omega\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t;L^p(\mathbb{T}^N))$[1], $\forall p\in[1,\infty)$. On dit que u est une solution cinétique de (0.1) lorsque

$$u\in L^p(\Omega;L^\infty(0,T;L^p(\mathbb{T}^N))),\ \forall p\in[1,\infty),\qquad \sigma\,\nabla u\in L^2(\Omega\times[0,T];L^2(\mathbb{T}^N)),$$

et lorsqu'il existe une mesure cinétique $m\geq n_1$ $\mathbb{P}$-p.s., telle que $(f=\mathbf{1}_{u>\xi},m)$ vérifie (0.2) pour toute fonction test $\varphi\in C_c^\infty(\mathbb{T}^N\times[0,T]\times\mathbb{R})$.

Cette méthode fournit un bon cadre technique pour établir le caractère bien posé du problème ; en particulier on prouve le résultat suivant.

Theorem 0.0.1. *Soit $u_0\in L^p(\Omega;L^p(\mathbb{T}^N))$, pour tout $p\in[1,\infty)$. Sous les hypothèses de Chapitre 3, il existe une unique solution cinétique de (0.1) dont les trajectoires sont presque sûrement continues à valeurs dans $L^p(\mathbb{T}^N)$, pour tout $p\in[1,\infty)$. De plus, si u_1, u_2 sont des solutions cinétiques de (0.1) avec conditions initiales respectives $u_{1,0}$ et $u_{2,0}$ alors pour tout $t\in[0,T]$*

$$\mathbb{E}\|u_1(t)-u_2(t)\|_{L^1(\mathbb{T}^N)}\leq\mathbb{E}\|u_{1,0}-u_{2,0}\|_{L^1(\mathbb{T}^N)}.$$

Dans un travail préliminaire qui permet de montrer l'existence de solutions régulières pour des problèmes approchés apparaissant dans la preuve d'existence de (0.1), nous étudions des EDPS dirigées par un processus de Wiener d-dimensionnel de la forme :

$$\begin{aligned}\mathrm{d}u&=\big[\mathcal{A}u+F(u)\big]\mathrm{d}t+\sigma(u)\,\mathrm{d}W,\quad x\in\mathbb{T}^N,\ t\in(0,T),\\ u(0)&=u_0,\end{aligned}\tag{0.3}$$

où $-\mathcal{A}$ est un opérateur différentiel fortement elliptique d'ordre $2l$ avec coefficients variables, où le coefficient F est généralement un opérateur non-linéaire et non-borné et où σ est galement non borné. Dans le cas semilinéaire (0.3), la difficulté principale provient de la non-linéarité de F et σ. En effet, on ne peut pas dans ce cas appliquer l'argument de point fixe car, dans les espaces de Sobolev d'ordre supérieur, on ne peut espérer voir la condition Lipschitz satisfaite. Ce problème est étroitement lié aux propriétés fonctionnelles des opérateurs de Nemytskij, i.e. $T_G:h\mapsto G(h)$, où h appartient à un espace de fonctions E et $G:\mathbb{R}\to\mathbb{R}$ est non-linéaire. Il apparait que les propriétés fonctionnelles de ces opérateurs dépendent principalement du domaine de définition choisi. Même lorsque E est un espace de Sobolev, ils n'envoient pas nécéssairement E sur lui même (ces questions sont développées en détails dans le livre de Runst et Sickel [68]). Par exemple, si on considère $2\leq m\leq N/p$, $p\in(1,\infty)$, alors seuls les opérateurs linéaires envoient $W^{m,p}(\mathbb{T}^N)$ sur lui même. En revanche, pour tout $m\in\mathbb{N}$ et $p\in[1,\infty)$, et sous l'hypothèse que G est suffisamment régulière et a des dérivées bornées, on peut montrer que l'opérateur de Nemytskij T_G envoi $W^{1,mp}(\mathbb{T}^N)\cap W^{m,p}(\mathbb{T}^N)$ sur lui même et que pour tout $z\in W^{m,p}(\mathbb{T}^N)\cap W^{1,mp}(\mathbb{T}^N)$ on a

$$\|G(z)\|_{W^{m,p}(\mathbb{T}^N)}\leq C\big(1+\|z\|_{W^{m,p}(\mathbb{T}^N)}+\|z\|^m_{W^{1,mp}(\mathbb{T}^N)}\big).$$

C'est l'argument fondamental de notre preuve de régularité. Nous procédons par étapes successives. D'abord, nous étudions l'équation (0.3) dans $L^{mp}(\mathbb{T}^N)$ et appliquons le théorème du point fixe de Banach pour montrer l'existence d'une solution faible à valeurs dans $L^{mp}(\mathbb{T}^N)$. Ensuite, nous étudions ses itérations de Picard en tant que processus à valeurs dans l'espace de Sobolev $W^{1,mp}(\mathbb{T}^N)$. Sachant que T_G envoie $W^{1,mp}(\mathbb{T}^N)$ sur lui même, nous pouvons alors trouver une estimation uniforme au sens de la norme

[1] $\mathcal{P}$ la sigma-algèbre prévisible associée à $(\mathscr{F}_t)_{t\geq0}$.

$W^{1,mp}(\mathbb{T}^N)$ qui est utilisée ensuite pour trouver une estimation uniforme au sens de la norme $W^{m,p}(\mathbb{T}^N)$. Ces deux estimations restent valables pour le processus limite et, par conséquent, la solution faible est en fait forte. Le résultat final est le suivant.

Theorem 0.0.2. *Soit $p \in [2,\infty)$, $q \in (2,\infty)$, $m \in \mathbb{N}$. Supposons que*

$$u_0 \in L^q(\Omega; W^{m,p}(\mathbb{T}^N)) \cap L^{mq}(\Omega; W^{1,mp}(\mathbb{T}^N))$$

et

$$f_\alpha \in C^m(\mathbb{R}) \cap C^{2l-1}(\mathbb{R}),\ \ |\alpha| \leq 2l-1; \qquad \sigma_i \in C^m(\mathbb{T}^N \times \mathbb{R}),\ \ i = 1, \dots, d,$$

ont des dérivées bornées jusqu'à l'ordre m. Alors il existe une unique solution à (0.3) qui appartient à

$$L^q(\Omega; C([0,T]; W^{m,p}(\mathbb{T}^N))) \cap L^{mq}(\Omega; C([0,T]; W^{1,mp}(\mathbb{T}^N))),$$

et nous avons l'estimation

$$\begin{aligned}\mathbb{E} \sup_{0\leq t\leq T} \|u(t)\|^q_{W^{m,p}(\mathbb{T}^N)} + \mathbb{E} \sup_{0\leq t\leq T} \|u(t)\|^{mq}_{W^{1,mp}(\mathbb{T}^N)}& \\ \leq C\big(1 + \mathbb{E}\|u_0\|^q_{W^{m,p}(\mathbb{T}^N)} + \mathbb{E}\|u_0\|^{mq}_{W^{1,mp}(\mathbb{T}^N)}\big)&.\end{aligned}$$

Corollary 0.0.3. *Soit $k \in \mathbb{N}_0$ et $u_0 \in L^q(\Omega; C^{k+1}(\mathbb{T}^N))$ pour tout $q \in (2,\infty)$. Supposons que*

$$f_\alpha \in C^{k+1}(\mathbb{R}) \cap C^{2l-1}(\mathbb{R}),\ |\alpha| \leq 2l-1; \qquad \sigma_i \in C^{k+1}(\mathbb{T}^N \times \mathbb{R}),\ i = 1, \dots, d,$$

ont des dérivées bornées jusqu'à l'ordre $k+1$. Alors il existe une solution à (0.3) qui appartient à

$$L^q(\Omega; C([0,T]; C^{k,\lambda}(\mathbb{T}^N))) \qquad \textit{pour tout} \quad \lambda \in (0,1).$$

Dans la deuxième partie de cette thèse, on considère une loi de conservation hyperbolique avec un forçage aléatoire

$$\begin{aligned} \mathrm{d}u + \operatorname{div}\big(A(u)\big)\mathrm{d}t &= \varPhi(u)\,\mathrm{d}W, \qquad x \in \mathbb{T}^N,\ t \in (0,T), \\ u(0) &= u_0, \end{aligned} \tag{0.4}$$

et on étudie son approximation au sens de Bhatnagar-Gross-Krook (BGK). Dans ce cadre, on étend le résultat de Debussche et Vovelle [16] montrant le caractère bien posé de (0.4) au sens des solutions cinétiques. En particulier, on montre que la solution cinétique est limite macroscopique du modèle BGK stochastique lorsque l'échelle microscopique tend vers 0.

La motivation initiale vient de l'équivalent déterministe, qui a été très étudié dans la littérature (voir par exemple Berthelin et Vovelle [7], Imbert et Vovelle [38], Lions, Perthame et Tadmor [55], [56], Perthame [64], Perthame et Tadmor [65]). Dans ce cas, le modèle BGK est donné par

$$\big(\partial_t + a(\xi)\cdot\nabla\big) f^\varepsilon = \frac{\chi_{u^\varepsilon} - f^\varepsilon}{\varepsilon}, \qquad t > 0,\ x \in \mathbb{T}^N,\ \xi \in \mathbb{R}, \tag{0.5}$$

où la fonction déquilibre χ_{u^ε} est définie par

$$\chi_{u^\varepsilon}(\xi) = \mathbf{1}_{0<\xi<u^\varepsilon} - \mathbf{1}_{u^\varepsilon<\xi<0},$$

a étant la dérivée de A, et la densité locale de particules étant définie par

$$u^\varepsilon(t,x) = \int_{\mathbb{R}} f^\varepsilon(t,x,\xi)\,\mathrm{d}\xi.$$

L'idée est la suivante : quand $\varepsilon \to 0$, les solutions f^ε de (0.5) convergent vers χ_u, où u est l'unique solution cinétique ou entropique de la loi de conservation scalaire déterministe.

Dans le cas stochastique, le modèle BGK s'écrit

$$\begin{aligned} \mathrm{d}F^\varepsilon + a(\xi)\cdot\nabla F^\varepsilon\,\mathrm{d}t &= \frac{\mathbf{1}_{u^\varepsilon>\xi} - F^\varepsilon}{\varepsilon}\,\mathrm{d}t - \partial_\xi F^\varepsilon \varPhi\,\mathrm{d}W - \frac{1}{2}\partial_\xi\big(G^2(-\partial_\xi F^\varepsilon)\big)\,\mathrm{d}t, \\ F^\varepsilon(0) &= F_0^\varepsilon, \end{aligned} \tag{0.6}$$

où la fonction F^ε correspond à $f^\varepsilon + \mathbf{1}_{0>\xi}$ et la densité locale u^ε est donnée ci-dessus. Une solution du modèle BGK stochastique (0.6) est considérée au sens faible : un processus prévisible $F^\varepsilon \in L^\infty(\Omega\times[0,T]\times\mathbb{T}^N\times\mathbb{R})$ tel que $F^\varepsilon - \mathbf{1}_{0>\xi} \in L^1(\Omega\times[0,T]\times\mathbb{T}^N\times\mathbb{R})$ est appelé solution faible de (0.6) s'il satisfait (0.6) au sens des distributions dans $\mathcal{D}'(\mathbb{T}^N\times\mathbb{R})$ pour presque tout $t\in[0,T]$, $\mathbb{P}$-presque sûrement. On obtient le résultat suivant.

Theorem 0.0.4. *Sous les hypothèses de Chapitre 4, pour tout $\varepsilon > 0$, il existe $F^\varepsilon \in L^\infty(\Omega\times[0,T]\times\mathbb{T}^N\times\mathbb{R})$ unique solution faible du modèle BGK stochastique (0.6) avec condition initiale $F_0^\varepsilon = \mathbf{1}_{u_0^\varepsilon>\xi}$. De plus, si $f^\varepsilon = F^\varepsilon - \mathbf{1}_{0>\xi}$ alors (f^ε) converge dans $L^p(\Omega\times[0,T]\times\mathbb{T}^N\times\mathbb{R})$, pour tout $p\in[1,\infty)$, vers la fonction d'équilibre χ_u, où u est l'unique solution cinétique de la loi de conservation hyperbolique stochastique (0.4). En outre, les densités locales (u^ε) convergent vers la solution cinétique u dans $L^p(\Omega\times[0,T]\times\mathbb{T}^N)$, pour tout $p\in[1,\infty)$.*

Dans la dernière partie de cette thèse, on donne une preuve nouvelle et très élémentaire du théorème classique dû à Skorokhod (voir [71], [72]) concernant l'existence de solutions faibles d'équations différentielles stochastiques

$$\mathrm{d}X = b(t,X)\,\mathrm{d}t + \sigma(t,X)\,\mathrm{d}W, \qquad X(0) \overset{d}{\sim} \nu. \tag{0.7}$$

où les coefficients sont des fonctions boréliennes, continues en la deuxième variable. Dans un premier temps, on impose une condition de croissance linéaire, qui est ensuite relaxée et remplacée par une condition de Lyapunov appropriée.

La preuve classique repose sur deux outils non triviaux : le théorème de représentation de Skorokhod et le théorème de représentation intégrale, dont la preuve devient très technique lorsque la dimension spatiale est supérieure à 1. Une approche alternative permettant l'identification de la limite a été découverte récemment par Ondreját [11], [60], concernant l'étude des fonctions d'ondes stochastiques entre variétés, lorsque le théorème de représentation intégrale des martingales n'est plus valide. On franchit une étape supplémentaire, en s'affranchissant du théorème de représentation de Skorokhod. L'argument est inspiré de preuves tirées de Jacod et Shiryaev [41], et la démonstration que nous proposons n'est pas difficile et presque entièrement autonome ; elle nécessite seulement deux lemmes auxiliaires dont la preuve est simple. On obtient le résultat suivant.

Theorem 0.0.5. *Soit $b:[0,T]\times\mathbb{R}^m\to\mathbb{R}^m$ et $\sigma:[0,T]\times\mathbb{R}^m\to\mathbb{M}_{m\times n}$ deux fonctions boréliennes telles que $b(t,\cdot)$ et $\sigma(t,\cdot)$ sont continues sur $\mathbb{R}^m$ pour tout $t\in[0,T]$, et telles que l'hypothèse de croissance linéaire est satisfaite :*

$$\exists K_* < \infty\ \forall t\in[0,T]\ \forall x\in\mathbb{R}^m \quad \|b(t,x)\|\vee\|\sigma(t,x)\| \le K_*\big(1+\|x\|\big).$$

Soit ν une mesure de probabilité de Borel sur $\mathbb{R}^m$. Alors le problème (0.7) admet une solution faible.

De plus, on voit que cette nouvelle méthode s'applique également si on relaxe l'hypothèse de condition linéaire, en demandant l'existence d'une fonction de Lyapunov appropriée.

Theorem 0.0.6. *Supposons que l'hypothèse*

(A) *$b(r,\cdot)$ et $\sigma(r,\cdot)$ sont continues sur $\mathbb{R}^m$ pour tout $r \in [0,T]$ et les fonctions b et σ sont localement bornées sur $[0,T] \times \mathbb{R}^m$, i.e.*

$$\sup_{r\in[0,T]} \sup_{\|z\|\leq L} \big\{\|b(r,z)\| \vee \|\sigma(r,z)\|\big\} < \infty$$

pour tout $L \geq 0$,

est satisfaite, et qu'il existe une fonction $V \in \mathscr{C}^2(\mathbb{R}^m)$ telle que

(L1) *il existe une fonction croissante $\kappa\colon \mathbb{R}_+ \to]0,\infty[$ telle que $\lim_{r\to\infty} \kappa(r) = +\infty$ et $V(x) \geq \kappa(\|x\|)$ pour tout $x \in \mathbb{R}^m$,*

(L2) *il existe $\gamma \geq 0$ tel que*

$$\big\langle b(t,x), DV(x)\big\rangle + \frac{1}{2}\operatorname{Tr}\big(\sigma(t,x)^* D^2V(x)\sigma(t,x)\big) \leq \gamma V(x)$$

pour tout $(t,x) \in [0,T] \times \mathbb{R}^m$.

Alors (0.7) admet une solution faible.

Table of contents

Chapter 1

Introduction

This thesis contributes to the fields of stochastic partial differential equations (SPDEs) and stochastic differential equations (SDEs), dynamically developing subjects that lie at the cross section of probability theory, the theory of partial differential equations (PDEs) and mathematical physics. PDEs are used to describe a wide variety of physical phenomena and from several points of view, depending on the concrete application, it is reasonable to add a stochastic noise. In such a way one obtains SPDEs. It is a well known fact in the field of PDEs and SPDEs that many equations do not, in general, have classical or strong solutions and can be solved only in some weaker sense. And therefore the very first question one has to ask while studying these models is: in which sense can we solve this equation? Nevertheless, too weak notion of solution often leads to existence of multiple solutions and uniqueness is lost. Hence it is necessary to find some balance that allows to establish existence of a unique (physically reasonable) solution. The third point we usually require is continuous dependence on initial data. Once these tasks are accomplished we can study further properties of the solution such as long time behavior or develop schemes for numerical simulations of our model.

In this thesis, we address several problems arising in the study of nondegenerate and degenerate parabolic SPDEs, hyperbolic conservation laws and SDEs with continuous coefficients. Let us now introduce the models to be studied and summarize the main results.

1.1 Degenerate parabolic SPDEs

This class of equations is the main object of study of Chapter 3. In particular, we consider the Cauchy problem for a scalar semilinear degenerate parabolic SPDE of the following form

$$
\begin{aligned}
\mathrm{d}u + \operatorname{div}\big(B(u)\big)\,\mathrm{d}t &= \operatorname{div}\big(A(x)\nabla u\big)\mathrm{d}t + \varPhi(u)\,\mathrm{d}W, \qquad x \in \mathbb{T}^N,\ t \in (0,T),\\
u(0) &= u_0,
\end{aligned}
\tag{1.1}
$$

where W is a cylindrical Wiener process. Equations of this type are used in fluid mechanics since they model the phenomenon of convection-diffusion of ideal fluid in porous media. Namely, the important applications including for instance two or three-phase flows can be found in petroleum engineering and hydrogeology. The addition of a stochastic noise to this physical model is fully natural as it represents external perturbations or a lack of knowledge of certain physical parameters. In order to get applicable results, it is necessary to treat the problem (1.1) under very general hypotheses. Particularly, the assumption of positive definiteness of the diffusion matrix A is not natural hence the equation can be degenerate which causes the main difficulty in solving the problem (1.1). We assume the matrix A to be positive semidefinite and, as a consequence, it can for instance vanish completely, which leads to a hyperbolic conservation law

$$
\begin{aligned}
\mathrm{d}u + \operatorname{div}\big(B(u)\big)\,\mathrm{d}t &= \varPhi(u)\,\mathrm{d}W,\\
u(0) &= u_0,
\end{aligned}
\tag{1.2}
$$

or it can only vanish on some subdomain of $\mathbb{T}^N$ for which we have no further assumptions. We point out, that we do not intend to employ any form of regularization by the noise to solve (1.1) and thus we include the deterministic equation in our theory as well. In order to find a suitable concept of solution, we observe that already in the case of deterministic hyperbolic conservation law

$$\begin{aligned}\partial_t u + \operatorname{div}\big(B(u)\big) &= 0,\\ u(0) &= u_0,\end{aligned}$$

it is possible to find simple examples supporting the two following claims (see e.g. [57]):

(i) classical C^1 solutions do not exist,

(ii) weak (distributional) solutions lack uniqueness.

The first claim is a consequence of the fact that any smooth solution has to be constant along characteristics but these can intersect in finite time (even in the case of smooth data) and shocks can be produced. The second claim demonstrates the inconvenience that was already indicated above: the usual way of weakening the equation leads to occurrence of nonphysical solutions and therefore additional assumptions need to be imposed in order to select the physically relevant ones. Although there exist several possible ways in the literature, we adapt the notion of kinetic formulation and kinetic solution. This concept that was first introduced by Lions, Perthame, Tadmor [56] for deterministic hyperbolic conservation laws and applies to more general situations than the one of entropy solution as introduced by Kružkov [45] (we refer the reader to Chapter 3 for further references). Moreover, it appears to be better suited particularly for degenerate parabolic problems since it allows us to keep the precise structure of the parabolic dissipative measure, whereas in the case of entropy solution part of this information is lost and has to be recovered at some stage. This technique also supplies a good technical framework to establish the well-posedness theory.

Among other significant references in this direction, let us emphasize the paper of Chen and Perthame [13] who studied the case of deterministic degenerate parabolic PDE of the form

$$\begin{aligned}\partial_t u + \operatorname{div}\big(B(u)\big) &= \operatorname{div}\big(A(u)\nabla u\big),\\ u(0) &= u_0,\end{aligned}\tag{1.3}$$

by means of both entropy and kinetic solutions. The first work dealing with kinetic solutions in the stochastic setting and also the first complete well-posedness result for hyperbolic conservation laws driven by a general multiplicative noise (1.2) was given by Debussche and Vovelle [16]. In comparison to this case, i.e. equation (1.1) with $A = 0$, the problem (1.1) is significantly more difficult. Indeed, Debussche and Vovelle defined a notion of generalized kinetic solution and obtained a comparison result showing that any generalized kinetic solution is actually a kinetic solution. Accordingly, the proof of existence simplified since only weak convergence of approximate viscous solutions was necessary. The situation is quite different in the case of (1.1) as we are not able to apply this approach: we prove the comparison principle only for kinetic solutions (not generalized ones) and therefore strong convergence of approximate solutions is needed in order to prove the existence. Moreover, the proof of the comparison principle itself is much more delicate as it was necessary to develop a suitable method to control the parabolic term.

The study of well-posedness for quasilinear degenerate parabolic SPDE's

$$\begin{aligned}\mathrm{d}u + \operatorname{div}\big(B(u)\big)\,\mathrm{d}t &= \operatorname{div}\big(A(u)\nabla u\big)\,\mathrm{d}t + \varPhi(u)\,\mathrm{d}W,\\ u(0) &= u_0,\end{aligned}$$

is in progress. Due to a recent result concerning a generalized Itô formula, the corresponding kinetic formulation might be derived also for weak solutions to suitable nondegenerate approximations hence the necessity of regular approximate solutions might be avoided. Indeed, the question of regularity in this case is interesting but highly delicate. Even in the deterministic setting (1.3) the proofs that can be found in [51] are very difficult and technical while the stochastic case still remains open.

1.1.1 Kinetic solutions

As already mentioned above, the basic idea for establishing well-posedness of (1.1) is to search for a criterion that

- ensures uniqueness,
- selects the physical solution,
- is fulfilled by any sufficiently smooth solution,

i.e. we need to weaken the problem (1.1) is some more efficient way. Towards this end, let us briefly review our hypotheses: let $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq0},\mathbb{P})$ be a stochastic basis; the flux function B is of class C^1 with a polynomial growth of its derivative b; the diffusion matrix A is of class C^∞, symmetric, positive semidefinite; the process W is a cylindrical Wiener process, i.e. $W(t)=\sum_{k\geq1}\beta_k(t)e_k$ with $(\beta_k)_{k\geq1}$ being mutually independent real-valued standard Wiener processes relative to $(\mathscr{F}_t)_{t\geq0}$ and $(e_k)_{k\geq1}$ a complete orthonormal system in a separable Hilbert space $\mathfrak{U}$; the mapping $\varPhi(z):\mathfrak{U}\to L^2(\mathbb{T}^N)$ is defined for each $z\in L^2(\mathbb{T}^N)$ by $\varPhi(z)e_k=g_k(\cdot,z(\cdot))$ where $g_k\in C(\mathbb{T}^N\times\mathbb{R})$ and the following conditions

$$G^2(x,\xi)=\sum_{k\geq1}|g_k(x,\xi)|^2\leq C\big(1+|\xi|^2\big),$$

$$\sum_{k\geq1}|g_k(x,\xi)-g_k(y,\zeta)|^2\leq C\big(|x-y|^2+|\xi-\zeta|h(|\xi-\zeta|)\big),$$

are fulfilled for every $x,y\in\mathbb{T}^N$, $\xi,\zeta\in\mathbb{R}$, with h being a continuous nondecreasing function on $\mathbb{R}_+$ satisfying, for some $\alpha>0$,

$$h(\delta)\leq C\delta^\alpha,\quad \delta<1. \tag{1.4}$$

With this in hand, let us consider a smooth solution to (1.1), namely, we assume $u\in C([0,T];C^2(\mathbb{T}^N))$ $\mathbb{P}$-a.s., so that (1.1) is satisfied pointwise

$$\begin{aligned}u(x,t)=u_0(x)&-\int_0^t\operatorname{div}\big(B(u(x,s))\big)\,\mathrm{d}s+\int_0^t\operatorname{div}\big(A(x)\nabla u(x,s)\big)\,\mathrm{d}s\\&+\sum_{k\geq1}\int_0^t g_k\big(x,u(x,s)\big)\,\mathrm{d}\beta_k(s),\quad \text{a.e. }(\omega,x)\in\Omega\times\mathbb{T}^N,\ \forall t\in[0,T].\end{aligned}$$

Then it is correct to apply the Itô formula to $\langle\mathbf{1}_{u(x,t)>\xi},\theta'\rangle_\xi=\theta(u(x,t))$[1], where $\theta\in C^\infty(\mathbb{R})$. Furthermore, it is possible to allow test functions that depend on x and t as

[1] $\langle\cdot,\cdot\rangle_\xi$ denotes the duality between distributions and test functions over $\mathbb{R}$.

well, i.e. $\varphi\in C_c^\infty(\mathbb{T}^N\times[0,T)\times\mathbb{R})$. In particular, we obtain[2]

$$\begin{aligned}\int_0^T\big\langle f(t),\partial_t\varphi(t)\big\rangle\mathrm{d}t+\big\langle f_0,\varphi(0)\big\rangle+\int_0^T\big\langle f(t),b(\xi)\cdot\nabla\varphi(t)\big\rangle\mathrm{d}t\\ +\int_0^T\big\langle f(t),\operatorname{div}\big(A(x)\nabla\varphi(t)\big)\big\rangle\mathrm{d}t\\ =-\sum_{k\geq1}\int_0^T\int_{\mathbb{T}^N}g_k\big(x,u(x,t)\big)\varphi\big(x,t,u(x,t)\big)\mathrm{d}x\,\mathrm{d}\beta_k(t)\\ -\frac{1}{2}\int_0^T\int_{\mathbb{T}^N}G^2\big(x,u(x,t)\big)\partial_\xi\varphi\big(x,t,u(x,t)\big)\mathrm{d}x\,\mathrm{d}t+n_1(\partial_\xi\varphi),\end{aligned}\tag{1.5}$$

where $f(x,t,\xi)=\mathbf{1}_{u(x,t)>\xi}$, $f_0=\mathbf{1}_{u_0>\xi}$ and n_1 is the parabolic dissipative measure defined by

$$\mathrm{d}n_1(x,t,\xi)=|\sigma(x)\nabla u|^2\,\mathrm{d}\delta_{u(x,t)}(\xi)\,\mathrm{d}x\,\mathrm{d}t,$$

where σ is the square-root matrix of A. Hence we have derived the kinetic formulation of (1.1) which formally reads

$$\partial_t f+b(\xi)\cdot\nabla f-\operatorname{div}\big(A(x)\nabla f\big)=\delta_{u=\xi}\,\varPhi(\xi)\dot{W}+\partial_\xi\Big(m-\frac{1}{2}G^2(x,\xi)\delta_{u=\xi}\Big)\tag{1.6}$$

and is solved by the pair (f,m) with m being a kinetic measure, i.e. a random nonnegative bounded Borel measure on $\mathbb{T}^N\times[0,T]\times\mathbb{R}$ that vanishes for large ξ in the following sense: if $B_R^c=\{\xi\in\mathbb{R};\,|\xi|\geq R\}$ then

$$\lim_{R\to\infty}\mathbb{E}\,m\big(\mathbb{T}^N\times[0,T]\times B_R^c\big)=0,$$

and consists of two components $m=n_1+n_2$, the parabolic dissipative measure n_1 and an unknown measure n_2 which takes account of possible singularities and vanishes in the nondegenerate case. A kinetic solution is then defined as follows: let $u\in L^p(\Omega\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t;L^p(\mathbb{T}^N))$[3], for all $p\in[1,\infty)$. It is said to be a kinetic solution to (1.1) provided

$$u\in L^p(\Omega;L^\infty(0,T;L^p(\mathbb{T}^N))),\ \forall p\in[1,\infty),\qquad\sigma\,\nabla u\in L^2(\Omega\times[0,T];L^2(\mathbb{T}^N)),$$

and there exists a kinetic measure $m\geq n_1$ $\mathbb{P}$-a.s. such that $(f=\mathbf{1}_{u>\xi},m)$ satisfies (1.5) $\mathbb{P}$-a.s. for any $\varphi\in C_c^\infty(\mathbb{T}^N\times[0,T]\times\mathbb{R})$.

A few remarks are in place. First, a kinetic solution is a class of equivalence in $L^p(\Omega\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t;L^p(\mathbb{T}^N))$ so it is not a stochastic process in the classical sense. However, it is shown in the very first step of the proof of uniqueness that, in this class of equivalence, there exists a representative that is a continuous $L^p(\mathbb{T}^N)$-valued stochastic process. Second, one very important feature of the kinetic formulation (1.6) is its linearity in f and, as a consequence, methods for linear equations can be applied. For instance, one can easily consider approximations of coefficients as only weak convergence of the approximate solutions, say f^n, is sufficient in order to pass to the limit. To summarize, the original equation (1.1), which is nonlinear in u, is transformed into (1.6) which is a linear equation of the nonlinear function $f=\mathbf{1}_{u>\xi}$.

Let us now formulate our well-posedness result.

[2] $\langle\cdot,\cdot\rangle$ denotes the duality between distributions and test functions over $\mathbb{T}^N\times\mathbb{R}$.

[3] $\mathcal{P}$ denotes the predictable σ-algebra associated to $(\mathscr{F}_t)_{t\geq0}$.

Theorem 1.1.1. *Let $u_0 \in L^p(\Omega; L^p(\mathbb{T}^N))$, for all $p \in [1,\infty)$. Under the above assumptions, there exists a unique kinetic solution to the problem (1.1) and it has almost surely continuous trajectories in $L^p(\mathbb{T}^N)$, for all $p \in [1,\infty)$. Moreover, if u_1, u_2 are kinetic solutions to (1.1) with initial data $u_{1,0}$ and $u_{2,0}$, respectively, then for all $t \in [0,T]$*

$$\mathbb{E}\|u_1(t) - u_2(t)\|_{L^1(\mathbb{T}^N)} \leq \mathbb{E}\|u_{1,0} - u_{2,0}\|_{L^1(\mathbb{T}^N)}. \tag{1.7}$$

The expression (1.7) is the L^1-comparison principle and is to be understood as a formula for the corresponding time-continuous representatives of u_1 and u_2. It yields pathwise uniqueness as well as continuous dependence on initial data in $L^1(\mathbb{T}^N)$.

1.1.2 Comparison principle

Since a kinetic solution u is a class of equivalence in $L^p(\Omega \times [0,T], \mathcal{P}, \mathrm{d}\mathbb{P}\otimes \mathrm{d}t; L^p(\mathbb{T}^N))$ and similarly also $f = \mathbf{1}_{u>\xi} \in L^\infty(\Omega \times \mathbb{T}^N \times [0,T] \times \mathbb{R})$, it is necessary to find suitable representatives in t (that are classes of equivalence in the remaining variables) with good continuity properties. In particular, f admits representatives f^-, f^+ which are $\mathbb{P}$-a.s. left- and right-continuous, respectively, on $[0,T]$ in the sense of $\mathcal{D}'(\mathbb{T}^N \times \mathbb{R})$. Furthermore, there exist $u^\pm : \Omega \times \mathbb{T}^N \times [0,T] \to \mathbb{R}$ such that $f^\pm = \mathbf{1}_{u^\pm>\xi}$ for a.e (ω, x, ξ) and all t and consequently $u^+ = u^- = u$ for a.e. $t \in [0,T]$. The representative u^+ is then shown to have almost surely continuous trajectories in $L^p(\mathbb{T}^N)$.

With this in hand, the weak formulation (1.5) can be strengthen to become only weak in x and ξ and the following result relating two kinetic solutions can be proved: Let u_1, u_2 be two kinetic solutions to (1.1) with initial data $u_{1,0}$, $u_{2,0}$, respectively, and denote $f_i = \mathbf{1}_{u_i>\xi}$, $f_{i,0} = \mathbf{1}_{u_{i,0}>\xi}$, $i = 1,2$. Then for $t \in [0,T]$ and any nonnegative functions $\varrho \in C^\infty(\mathbb{T}^N)$, $\psi \in C_c^\infty(\mathbb{R})$ we have

$$\begin{aligned}
&\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \varrho(x-y)\psi(\xi-\zeta) f_1^\pm(x,t,\xi)\bar{f}_2^\pm(y,t,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\leq \mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \varrho(x-y)\psi(\xi-\zeta) f_{1,0}(x,\xi)\bar{f}_{2,0}(y,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y + \mathrm{I} + \mathrm{J} + \mathrm{K},
\end{aligned}$$

where $\bar{f}$ denotes the conjugate function $\bar{f} = 1 - f$. For the precise definition of the remainders I, J, K we refer the reader to Proposition 3.3.2. Testing now by (ϱ_τ) and (ψ_δ), approximations to the identity on $\mathbb{T}^N$ and $\mathbb{R}$, respectively, it is possible to control these remainders and deduce that

$$\mathbb{E}\int_{\mathbb{T}^N}\int_{\mathbb{R}} f_1^\pm(t)\bar{f}_2^\pm(t)\,\mathrm{d}\xi\,\mathrm{d}x \leq \mathbb{E}\int_{\mathbb{T}^N}\int_{\mathbb{R}} f_{1,0}\bar{f}_{2,0}\,\mathrm{d}\xi\,\mathrm{d}x$$

which yields the comparison principle (1.7). Note that especially the term J that is obtained from the second order term of (1.1) requires a very fine analysis.

1.1.3 Existence

The proof is divided into two parts. In the first part, we prove the result under an additional assumption upon the initial condition: $u_0 \in L^p(\Omega; C^\infty(\mathbb{T}^N))$, for all $p \in [1,\infty)$. We employ the vanishing viscosity method: we approximate (1.1) by certain nondegenerate problems and consequent passage to the limit gives the existence of a kinetic solution to the original problem. To be more precise, we consider a viscous approximation of (1.1)

$$\begin{aligned} \mathrm{d}u^\varepsilon+\operatorname{div}\big(B^\varepsilon(u^\varepsilon)\big)\,\mathrm{d}t&=\operatorname{div}\big(A(x)\nabla u^\varepsilon\big)\,\mathrm{d}t+\varepsilon\Delta u^\varepsilon\,\mathrm{d}t+\varPhi^\varepsilon(u^\varepsilon)\,\mathrm{d}W,\\ u^\varepsilon(0)&=u_0, \end{aligned} \tag{1.8}$$

where $\varPhi^\varepsilon$, B^ε are suitably chosen approximations of $\varPhi$ and B, respectively. According to [33] (see Chapter 2), for each $\varepsilon>0$ there exists a $C^\infty(\mathbb{T}^N)$-valued process which is the unique strong solution to (1.8). To conclude the compactness argument, we need to establish several estimates uniform in ε. Namely, we obtain the following[4]

$$\begin{aligned} \mathbb{E}\|u^\varepsilon\|^p_{L^\infty(0,T;L^p(\mathbb{T}^N))}&\leq C, && p\in[2,\infty),\\ \mathbb{E}\|u^\varepsilon\|^q_{C^\lambda([0,T];H^{-2}(\mathbb{T}^N))}&\leq C, && \lambda\in(0,1/2),\ q\in[2,\infty),\\ \sup_{0\leq t\leq T}\mathbb{E}\|u^\varepsilon(t)\|_{W^{s,1}(\mathbb{T}^N)}&\leq C, && s\in\Big(0,\min\Big\{\frac{\alpha}{\alpha+1},\frac12\Big\}\Big). \end{aligned}$$

By interpolation and an Aubin-Dubinskii type compact embedding theorem, we obtain tightness of the set of joint laws of $\{\mu^\varepsilon=\mathbb{P}\circ(u^\varepsilon,W)^{-1};\,\varepsilon\in(0,1)\}$ in the path space $\mathcal{X}=\mathcal{X}_u\times\mathcal{X}_W$, where

$$\mathcal{X}_u=\Big\{u\in L^2\big(0,T;L^2(\mathbb{T}^N)\big)\cap C\big([0,T];H^{-1}(\mathbb{T}^N)\big);\ \boldsymbol{\varrho}_0u\in L^2(\mathbb{T}^N)\Big\}$$

equipped with the norm

$$\|\cdot\|_{\mathcal{X}_u}=\|\cdot\|_{L^2(0,T;L^2(\mathbb{T}^N))}+\|\cdot\|_{C([0,T];H^{-1}(\mathbb{T}^N))}+\|\boldsymbol{\varrho}_0\cdot\|_{L^2(\mathbb{T}^N)}$$

and $\mathcal{X}_W=C([0,T];\mathfrak{U}_0)$. Here, $\boldsymbol{\varrho}_0$ is the operator of restriction to $\{0\}$, i.e. $\boldsymbol{\varrho}_0u=u_0$, and $\mathfrak{U}_0\supset\mathfrak{U}$ such that the embedding $\mathfrak{U}\hookrightarrow\mathfrak{U}_0$ is Hilbert-Schmidt. Passing to a weakly convergent subsequence μ^{ε_n} and denoting by μ the limit law we apply the Skorokhod representation theorem to infer the existence of random variables $(\tilde{u}^n,\tilde{W}^n)$, $n\in\mathbb{N}$, and $(\tilde{u},\tilde{W})$ defined on a probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$ such that the laws of $(\tilde{u}^n,\tilde{W}^n)$ and $(\tilde{u},\tilde{W})$ under $\tilde{\mathbb{P}}$ coincide with μ^{ε_n} and μ, respectively, and $(\tilde{u}^n,\tilde{W}^n)$ converges $\tilde{\mathbb{P}}$-almost surely to $(\tilde{u},\tilde{W})$ in the topology of $\mathcal{X}$. Identification of $(\tilde{u},\tilde{W})$ with a (martingale) kinetic solution is based on a new general method of constructing martingale solutions of SPDEs, that does not rely on any kind of martingale representation theorem and therefore holds independent interest especially in situations where these representation theorems are no longer available. First applications were already done in [11], [60] and, in the finite-dimensional case, also in [34], [35] (see Chapter 5). In the present work, this method is further generalized as the martingales to be dealt with are only defined for almost all times. Finally, we make use of the Gyöngy-Krylov characterization of convergence in probability: existence of a martingale kinetic solution together with pathwise uniqueness leads to the existence of a pathwise kinetic solution, i.e. kinetic solution defined on the original probability space.

The general case of $u_0\in L^p(\Omega;L^p(\mathbb{T}^N))$, for all $p\in[1,\infty)$, is a straightforward consequence of the previous part. Indeed, we approximate the initial condition by a sequence $(u_0^\varepsilon)\subset L^p(\Omega;C^\infty(\mathbb{T}^N))$ such that $u_0^\varepsilon\to u_0$ in $L^1(\Omega;L^1(\mathbb{T}^N))$. Due to the comparison principle (1.7), we deduce that there exists $u\in L^1(\Omega\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t;L^1(\mathbb{T}^N))$ such that

$$u^\varepsilon\longrightarrow u\quad\text{in}\quad L^1(\Omega\times[0,T],\mathcal{P},\mathrm{d}\mathbb{P}\otimes\mathrm{d}t;L^1(\mathbb{T}^N)),$$

[4] α was introduced in (1.4)

where u^ε are the kinetic solutions to (1.1) with initial data u_0^ε. Accordingly, we conclude by passing to the limit in (1.5).

1.2 Regularity for the nondegenerate case

In this section, we present the results of Chapter 2. The aim is to ensure existence of smooth solutions to the approximate problems (1.8) that arise in the proof of existence of (1.1). Nevertheless, since the final regularity result is based on properties of strongly elliptic operators, namely, the equivalence of the corresponding power scale with classical Sobolev spaces and semigroup arguments, generalization to more general higher order equations does not cause any additional problems. Let us consider the following semilinear SPDE driven by a d-dimensional Wiener process:

$$\begin{aligned} \mathrm{d}u &= \big[\mathcal{A}u + F(u)\big]\mathrm{d}t + \sigma(u)\,\mathrm{d}W, \quad x\in\mathbb{T}^N,\ t\in(0,T),\\ u(0) &= u_0, \end{aligned} \tag{1.9}$$

where $-\mathcal{A}$ is a strongly elliptic differential operator of order $2l$ with variable coefficients of class $C^\infty(\mathbb{T}^N)$. Let us assume, in addition, that $-\mathcal{A}$ is formally symmetric and positive, i.e. 0 belongs to the resolvent set of $-\mathcal{A}$. The coefficient F is generally nonlinear unbounded operator defined as follows: for any $p\in[2,\infty)$

$$\begin{aligned} F: L^p(\mathbb{T}^N) &\longrightarrow W^{-2l+1,p}(\mathbb{T}^N)\\ z &\longmapsto \sum_{|\alpha|\le 2l-1} a_\alpha\,\mathrm{D}^\alpha f_\alpha(z), \end{aligned}$$

where $a_\alpha\in\mathbb{R}$ and the functions f_α, $|\alpha|\le 2l-1$, are smooth enough. The diffusion coefficient $\sigma(z):\mathbb{R}^d\to L^p(\mathbb{T}^N)$ is also nonlinear, defined for any $z\in L^p(\mathbb{T}^N)$ by $\sigma(z)e_k = \sigma_i(\cdot,z(\cdot))$[5] where the functions $\sigma_1,\dots,\sigma_d:\mathbb{T}^N\times\mathbb{R}\to\mathbb{R}$ are of linear growth.

As mentioned above, it is a common problem in the field of PDEs and SPDEs that many real-world problems do not admit classical or strong solutions and can be solved only in some weaker sense. For this reason the question of regularity is an interesting topic that does not always possess a satisfactory (affirmative) answer. Unlike deterministic problems, in the case of SPDEs we can only ask whether the solutions is smooth in the space variable. The main difficulty in the semilinear case (1.9) lies in the nonlinearities F and σ as, in higher order Sobolev spaces, we cannot expect the Lipschitz condition to be satisfied and hence the fixed point argument cannot be applied. This issue is closely related to the mapping properties of Nemytskij operators, i.e. $T_G: h\mapsto G(h)$, where h belongs to some function space E and $G:\mathbb{R}\to\mathbb{R}$ is nonlinear. It turns out (and was discussed in-depth in the book of Runst and Sickel [68]) that the mapping properties of these operators depend strongly on the chosen domain of definition and even for E being a Sobolev space they do not, in general, map E to itself. For example, if we consider $2\le m\le N/p$, $p\in(1,\infty)$, then only linear operators map $W^{m,p}(\mathbb{T}^N)$ to itself (see [68, Theorem 5.2.4/2]). On the other hand, for any $m\in\mathbb{N}$ and $p\in[1,\infty)$, under the hypothesis of a sufficiently smooth function G having bounded derivatives one arrives at the fact that the Nemytskij operator T_G maps $W^{1,mp}(\mathbb{T}^N)\cap W^{m,p}(\mathbb{T}^N)$ to itself and the following estimate holds true for any $z\in W^{m,p}(\mathbb{T}^N)\cap W^{1,mp}(\mathbb{T}^N)$

$$\|G(z)\|_{W^{m,p}(\mathbb{T}^N)} \le C\big(1+\|z\|_{W^{m,p}(\mathbb{T}^N)} + \|z\|^m_{W^{1,mp}(\mathbb{T}^N)}\big).$$

[5] $(e_k)_{k=1}^d$ is an orthonormal basis in $\mathbb{R}^d$.

It turns out to be the keystone of our proof of regularity. In particular, we proceed successively in several steps. First of all, we consider the equation (1.9) in $L^{mp}(\mathbb{T}^N)$ and apply the Banach fixed point theorem to conclude the existence of an $L^{mp}(\mathbb{T}^N)$-valued mild solution. Next, we study its Picard iterations as processes having values in the Sobolev spaces $W^{1,mp}(\mathbb{T}^N)$. Having known that T_G maps $W^{1,mp}(\mathbb{T}^N)$ to itself we are able to find a uniform estimate of the $W^{1,mp}(\mathbb{T}^N)$-norm which is then used in the last step to deduce a uniform estimate of the $W^{m,p}(\mathbb{T}^N)$-norm. Both estimates remain valid also for the limit process and, as a consequence, the mild solution is even strong. We obtain

Theorem 1.2.1. *Let $p \in [2, \infty)$, $q \in (2, \infty)$, $m \in \mathbb{N}$. We suppose that*

$$u_0 \in L^q(\Omega; W^{m,p}(\mathbb{T}^N)) \cap L^{mq}(\Omega; W^{1,mp}(\mathbb{T}^N))$$

and

$$f_\alpha \in C^m(\mathbb{R}) \cap C^{2l-1}(\mathbb{R}), \ \ |\alpha| \le 2l-1; \qquad \sigma_i \in C^m(\mathbb{T}^N \times \mathbb{R}), \ \ i = 1, \dots, d,$$

have bounded derivatives up to order m. Then there exists a unique solution to (1.9) which belongs to

$$L^q(\Omega; C([0,T]; W^{m,p}(\mathbb{T}^N))) \cap L^{mq}(\Omega; C([0,T]; W^{1,mp}(\mathbb{T}^N)))$$

and the following estimate holds true

$$\begin{aligned} \mathbb{E} \sup_{0\le t\le T} \|u(t)\|^q_{W^{m,p}(\mathbb{T}^N)} + \mathbb{E} \sup_{0\le t\le T} \|u(t)\|^{mq}_{W^{1,mp}(\mathbb{T}^N)} & \\ \le C\big(1 + \mathbb{E}\|u_0\|^q_{W^{m,p}(\mathbb{T}^N)} &+ \mathbb{E}\|u_0\|^{mq}_{W^{1,mp}(\mathbb{T}^N)}\big). \end{aligned}$$

Corollary 1.2.2. *Let $k \in \mathbb{N}_0$ and $u_0 \in L^q(\Omega; C^{k+1}(\mathbb{T}^N))$ for all $q \in (2, \infty)$. Assume that*

$$f_\alpha \in C^{k+1}(\mathbb{R}) \cap C^{2l-1}(\mathbb{R}), \ |\alpha| \le 2l-1; \qquad \sigma_i \in C^{k+1}(\mathbb{T}^N \times \mathbb{R}), \ i = 1, \dots, d,$$

have bounded derivatives up to order $k+1$. Then there exists a solution to (1.9) which belongs to

$$L^q(\Omega; C([0,T]; C^{k,\lambda}(\mathbb{T}^N))) \qquad \textit{for every} \quad \lambda \in (0,1).$$

1.2.1 Proof

In order to solve the problem (1.9) in the $L^{mp}(\mathbb{T}^N)$-setting (and later also in $W^{1,mp}(\mathbb{T}^N)$ and $W^{m,p}(\mathbb{T}^N)$) it was necessary to ensure the existence of the stochastic integral $\int_0^t \sigma(u)\,\mathrm{d}W$ in these spaces. As all of them belong to the class of the so-called 2-smooth Banach spaces, we made use of the stochastic Itô integration theory developed by Brzeźniak [10]. Let $\mathcal{S}$ be the strongly continuous analytic semigroup generated by $\mathcal{A}$. Then it is shown by the Banach fixed point theorem that (1.9) admits a unique mild solution

$$u(t) = \mathcal{S}(t)u_0 + \int_0^t \mathcal{S}(t-s)F(u(s))\,\mathrm{d}s + \int_0^t \mathcal{S}(t-s)\sigma(u(s))\,\mathrm{d}W(s)$$

that belongs to $L^q(\Omega \times [0,T], \mathcal{P}, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t; L^{mp}(\mathbb{T}^N))$. In order to obtain a better regularity of u, recall that it is the limit of Picard iterations: let $u^0(t) = u_0$ and for $n \in \mathbb{N}$ we define

$$u^n(t)=\mathcal{S}_p(t)\,u_0+\int_0^t\mathcal{S}_p(t-s)\,F\big(u^{n-1}(s)\big)\,\mathrm{d}s+\int_0^t\mathcal{S}_p(t-s)\,\sigma\big(u^{n-1}(s)\big)\,\mathrm{d}W(s).$$

By induction in n and by the fact that for any $z\in W^{1,mp}(\mathbb{T}^N)$

$$\|F(z)\|_{W^{1,mp}(\mathbb{T}^N)}+\|\sigma(z)\|_{W^{1,mp}(\mathbb{T}^N)}\leq C\big(1+\|z\|_{W^{1,mp}(\mathbb{T}^N)}\big),$$

we deduce

$$\mathbb{E}\sup_{0\leq t\leq T}\|u^n(t)\|^q_{W^{1,mp}(\mathbb{T}^N)}\leq C\big(1+\mathbb{E}\|u_0\|^q_{W^{1,mp}(\mathbb{T}^N)}\big),\qquad\forall n\in\mathbb{N},$$

with a constant C independent of n. As a consequence, we get the estimate

$$\mathbb{E}\sup_{0\leq t\leq T}\|u(t)\|^q_{W^{1,mp}(\mathbb{T}^N)}\leq C\big(1+\mathbb{E}\|u_0\|^q_{W^{1,mp}(\mathbb{T}^N)}\big)$$

hence the mild solution to (1.9) belongs to $L^q(\Omega;C([0,T];W^{1,mp}(\mathbb{T}^N)))$.

Proof of regularity in higher order Sobolev spaces (order greater than 1) is more complicated as the norm of a superposition does not, in general, grow linearly with the norm of the inner function. However, as for any $z\in W^{m,p}(\mathbb{T}^N)\cap W^{1,mp}(\mathbb{T}^N)$

$$\|F(z)\|_{W^{m,p}(\mathbb{T}^N)}+\|\sigma(z)\|_{W^{m,p}(\mathbb{T}^N)}\leq C\big(1+\|z\|_{W^{m,p}(\mathbb{T}^N)}+\|z\|^m_{W^{1,mp}(\mathbb{T}^N)}\big),$$

we can make use of the previous step to verify

$$\mathbb{E}\sup_{0\leq t\leq T}\|u^n(t)\|^q_{W^{m,p}(\mathbb{T}^N)}\leq C\big(1+\mathbb{E}\|u_0\|^q_{W^{m,p}(\mathbb{T}^N)}+\mathbb{E}\|u_0\|^{mq}_{W^{1,mp}(\mathbb{T}^N)}\big),$$

where the constant is independent of n, hence

$$\mathbb{E}\sup_{0\leq t\leq T}\|u(t)\|^q_{W^{m,p}(\mathbb{T}^N)}\leq C\big(1+\mathbb{E}\|u_0\|^q_{W^{m,p}(\mathbb{T}^N)}+\mathbb{E}\|u_0\|^{mq}_{W^{1,mp}(\mathbb{T}^N)}\big).$$

and the mild solution of (1.9) belongs to $L^q(\Omega;C([0,T];W^{m,p}(\mathbb{T}^N)))$. The Corollary 1.2.2 then follows from the Sobolev embedding theorem.

1.3 Stochastic Bhatnagar-Gross-Krook model

In Chapter 4, we consider a scalar conservation law with stochastic forcing

$$\begin{aligned}\mathrm{d}u+\mathrm{div}\,\big(A(u)\big)\mathrm{d}t&=\varPhi(u)\,\mathrm{d}W,\qquad t\in(0,T),\,x\in\mathbb{T}^N,\\u(0)&=u_0\end{aligned}\tag{1.10}$$

and study its approximation in the sense of Bhatnagar-Gross-Krook (a BGK-like approximation for short). In particular, we describe the conservation law (1.10) as the hydrodynamic limit of the stochastic BGK model, as the microscopic scale ε goes to 0. As the latter are much simpler equations that can be solved explicitly, this analysis can be used for developing innovative numerical schemes for hyperbolic conservation laws leading to practical applications in physics. In this sense, we extend the result of Debussche and Vovelle [16], who showed the well-posedness for kinetic solutions of (1.10). Note, that this is also covered by our result of Chapter 3 in the particular case $A=0$, however, as dealing with a general second order term brings many difficulties, Debussche and Vovelle [16] were able to prove a stronger result. In particular, they defined a notion of generalized kinetic solution and obtained a comparison principle that says that any generalized kinetic solution is a in fact a kinetic solution. Accordingly, we intend to show that the generalized kinetic solution (thus the kinetic solution) is the macroscopic limit of stochastic BGK approximations.

The initial motivation came from the deterministic counterpart that has already been extensively studied in the literature (see [7], [38], [55], [56], [58], [59], [65], [64]). In that case, the BGK model is given as follows

$$\big(\partial_t + a(\xi)\cdot\nabla\big) f^\varepsilon = \frac{\chi_{u^\varepsilon} - f^\varepsilon}{\varepsilon}, \qquad t>0,\, x\in\mathbb{T}^N,\, \xi\in\mathbb{R}, \tag{1.11}$$

where χ_{u^ε}, the so-called equilibrium function, is defined by

$$\chi_{u^\varepsilon}(\xi) = \mathbf{1}_{0<\xi<u^\varepsilon} - \mathbf{1}_{u^\varepsilon<\xi<0},$$

and a is the derivative of A. The differential operator ∇ is with respect to the space variable x. The additional real-valued variable ξ is called velocity; the solution f^ε is then a microscopic density of particles at (t,x) with velocity ξ. The local density of particles is defined by

$$u^\varepsilon(t,x) = \int_{\mathbb{R}} f^\varepsilon(t,x,\xi)\,\mathrm{d}\xi.$$

The collisions of particles are given by the nonlinear kernel on the right hand side of (1.11). The idea is that, as $\varepsilon\to 0$, the solutions f^ε of (1.11) converge to χ_u where u is the unique kinetic or entropy solution of the deterministic scalar conservation law. The BGK model in the stochastic case reads as

$$\begin{aligned} \mathrm{d}F^\varepsilon + a(\xi)\cdot\nabla F^\varepsilon\,\mathrm{d}t &= \frac{\mathbf{1}_{u^\varepsilon>\xi} - F^\varepsilon}{\varepsilon}\,\mathrm{d}t - \partial_\xi F^\varepsilon \varPhi\,\mathrm{d}W - \frac{1}{2}\partial_\xi\big(G^2(-\partial_\xi F^\varepsilon)\big)\,\mathrm{d}t,\\ F^\varepsilon(0) &= F_0^\varepsilon, \end{aligned} \tag{1.12}$$

where the function F^ε corresponds to $f^\varepsilon + \mathbf{1}_{0>\xi}$ and the local density u^ε is given as above. The general concept of the proof is as follows: First, the stochastic characteristics method developed by Kunita [50] is used to study certain auxiliary problem and existence of a unique solution to the stochastic BGK model is obtained for any fixed ε. Second, uniform estimates are established that together with the results of Debussche and Vovelle [16] justify the limit argument.

Let us make some comments on the deterministic BGK model (1.11). Even though the general concept of the proof is analogous, we point out that the techniques required by the stochastic case are significantly different. In particular, the characteristic system for the deterministic BGK model consists of independent equations

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = a_i(\xi), \qquad i=1,\dots,N,$$

and the ξ-coordinate of the characteristic curve is constant. Accordingly, it is much easier to control the behavior of f^ε for large ξ. Namely, if the initial data f_0^ε are compactly supported (in ξ), the same remains valid also for the solution itself and also the convergence proof simplifies. On the contrary, in the stochastic case, the ξ-coordinate of the characteristic curve is governed by an SDE and therefore this property is, in general, lost. Similar issues has to be dealt with in order to obtain all the necessary uniform estimates. To overcome this difficulty, it was needed to develop a suitable method to control the decay at infinity in connection with the remaining variables ω, t, x. Using this approach we are able to prove the convergence of the BGK model under a slightly weaker hypothesis on the initial datum u_0 than usually assumed in the deterministic case: it is not supposed to be bounded, we only assume $u_0 \in L^p(\Omega \times \mathbb{T}^N)$ for all $p \in [1,\infty)$. Note, that under this condition, the initial data for the deterministic BGK model, for instance $f_0^\varepsilon = \chi_{u_0}$, are not compactly supported and so the usual methods are not applicable. In the deterministic case, however, the boundedness assumption is fairly natural since also the solution u to the conservation law remains bounded. Obviously, this is not true for the stochastic case as it is impossible to get any L^∞_ω estimates due to the active white noise term.

There is another difficulty coming from the complex structure of the characteristic system for the stochastic BGK model (1.12). Namely, the finite speed of propagation that is an easy consequence of boundedness of the solution u of the conservation law in the deterministic case (see for instance [65]) is no longer valid and therefore some growth assumptions on the transport coefficient a are in place. The hypothesis of bounded derivatives is natural for the stochastic characteristics method as it implies the existence of global stochastic flows. Even though this already includes one important example of Burgers' equation it is of essential interest to handle also more general coefficients having polynomial growth. This was achieved by a suitable cut-off procedure which also guarantees all the necessary estimates.

Let us state the assumptions. The hypotheses in the paper of Debussche and Vovelle were the same as for the degenerate parabolic SPDEs (see Section 1.1 above), however, in view of the application of the stochastic characteristics method, these assumptions need to be strengthen. To be more precise, we assume the flux function A to be of class $C^{4,\eta}$, for some $\eta > 0$, with a polynomial growth of its first derivative a. The driving process W is a d-dimensional Wiener process and the diffusion coefficient $\varPhi(z) : \mathbb{R}^d \to L^2(\mathbb{T}^N)$ is again defined for any $z \in L^2(\mathbb{T}^N)$ by $\varPhi(z)e_k = g_k(\cdot, z(\cdot))$ where the functions $g_1, \dots, g_d$ are of class $C^{4,\eta}$ with linear growth and bounded derivatives of all orders. However, in order to get all the necessary estimates we restrict ourselves to two special cases: either

$$g_k(x,0) = 0, \qquad x \in \mathbb{T}^N,\ k = 1, \dots, d,$$

or

$$|g_k(x,\xi)| \le C, \qquad x \in \mathbb{T}^N,\ \xi \in \mathbb{R},\ k = 1, \dots, d.$$

Note, that the latter is satisfied for instance in the case of additive noise. Concerning the initial data for the BGK model (1.12), one possibility is to consider simply $F_0^\varepsilon = \mathbf{1}_{u_0>\xi}$, however, one can also take some suitable approximations of $\mathbf{1}_{u_0>\xi}$. Namely, let $\{u_0^\varepsilon;\ \varepsilon \in (0,1)\}$ be a set of approximate $\mathscr{F}_0$-measurable initial data, which is bounded in $L^p(\Omega; L^p(\mathbb{T}^N))$ for all $p \in [1,\infty)$, and assume in addition that $u_0^\varepsilon \to u_0$ in $L^1(\Omega; L^1(\mathbb{T}^N))$.

Theorem 1.3.1 (Hydrodynamic limit of the stochastic BGK model)**.** *Let the above assumptions hold true. Then, for any $\varepsilon > 0$, there exists $F^\varepsilon \in L^\infty_{\mathcal{P}}(\Omega \times [0,T] \times \mathbb{T}^N \times \mathbb{R})$*[6] *which is a unique weak solution to the stochastic BGK model (1.12) with initial condition $F_0^\varepsilon = \mathbf{1}_{u_0^\varepsilon>\xi}$. Furthermore, if $f^\varepsilon = F^\varepsilon - \mathbf{1}_{0>\xi}$ then (f^ε) converges in $L^p(\Omega \times [0,T] \times \mathbb{T}^N \times \mathbb{R})$, for all $p \in [1,\infty)$, to the equilibrium function χ_u, where u is the unique kinetic solution to the stochastic hyperbolic conservation law (1.10). Besides, the local densities (u^ε) converge to the kinetic solution u in $L^p(\Omega \times [0,T] \times \mathbb{T}^N)$, for all $p \in [1,\infty)$.*

[6] i.e. F^ε is measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{T}^N) \otimes \mathcal{B}(\mathbb{R})$.

1.3.1 Existence

A solution to the stochastic BGK model (1.12) is understood in the weak sense: $F^\varepsilon \in L^\infty_{\mathcal{P}}(\Omega\times[0,T]\times\mathbb{T}^N\times\mathbb{R})$ satisfying $F^\varepsilon - \mathbf{1}_{0>\xi} \in L^1(\Omega\times[0,T]\times\mathbb{T}^N\times\mathbb{R})$ is called a weak solution to (1.12) provided for a.e. $t\in[0,T]$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\big\langle F^\varepsilon(t),\varphi\big\rangle = \big\langle F^\varepsilon_0,\varphi\big\rangle + \int_0^t \big\langle F^\varepsilon(s), a\cdot\nabla\varphi\big\rangle\,\mathrm{d}s\\ + \frac{1}{\varepsilon}\int_0^t \big\langle \mathbf{1}_{u^\varepsilon(t)>\xi} - F^\varepsilon(t),\varphi(t)\big\rangle\,\mathrm{d}t + \sum_{k=1}^d\int_0^t \big\langle F^\varepsilon(s),\partial_\xi(g_k\varphi)\big\rangle\,\mathrm{d}\beta_k(s)\\ + \frac{1}{2}\int_0^t \big\langle F^\varepsilon(s),\partial_\xi(G^2\partial_\xi\varphi)\big\rangle\,\mathrm{d}s.\end{aligned}$$

We intend to employ the stochastic characterics method hence it is more natural to work with the Stratonovich integral as the Itô-Wentzell-type formula is then close to the classical differential rule formula for composite functions. It can be seen that on the level of the above defined weak solutions the problem (1.12) is equivalent to

$$\begin{aligned}\mathrm{d}F^\varepsilon + a(\xi)\cdot\nabla F^\varepsilon\,\mathrm{d}t = \frac{\mathbf{1}_{u^\varepsilon>\xi} - F^\varepsilon}{\varepsilon}\,\mathrm{d}t - \partial_\xi F^\varepsilon \varPhi \circ \mathrm{d}W + \frac{1}{4}\partial_\xi F^\varepsilon\partial_\xi G^2\,\mathrm{d}t,\\ F^\varepsilon(0) = F^\varepsilon_0.\end{aligned}$$

Therefore, in the first step we study the auxiliary problem

$$\begin{aligned}\mathrm{d}X + a(\xi)\cdot\nabla X\,\mathrm{d}t = -\partial_\xi X\varPhi\circ\mathrm{d}W + \frac{1}{4}\partial_\xi X\partial_\xi G^2\,\mathrm{d}t.\\ X(s) = X_0.\end{aligned}\tag{1.13}$$

and show existence of a unique weak solution provided $X_0\in L^\infty(\Omega\times\mathbb{T}^N\times\mathbb{R})$. We define $\mathcal{S} = \{\mathcal{S}(t,s);\,0\le s\le t\le T\}$ to be the solution operator of (1.13). Having this in hand, it follows from the Duhamel principle that there exists a unique weak solution of the stochastic BGK model (1.12) and is represented by

$$F^\varepsilon(t) = \mathrm{e}^{-\frac{t}{\varepsilon}}\mathcal{S}(t,0)F^\varepsilon_0 + \frac{1}{\varepsilon}\int_0^t \mathrm{e}^{-\frac{t-s}{\varepsilon}}\mathcal{S}(t,s)\mathbf{1}_{u^\varepsilon(s)>\xi}\,\mathrm{d}s.\tag{1.14}$$

In order to solve (1.13) we proceed in two steps. The problem is written in the form that is suitable for the stochastic characteristics method, however, its coefficients are not supposed to have bounded derivatives hence the existence of global solutions is not guaranteed. To overcome this difficulty, we first employ truncations and then pass to the limit. Let us consider

$$\begin{aligned}\mathrm{d}X + a^R(\xi)\cdot\nabla X\,\mathrm{d}t = -\partial_\xi X\varPhi^R\circ\mathrm{d}W + \frac{1}{4}\partial_\xi X\partial_\xi G^{R,2}\,\mathrm{d}t,\\ X(s) = X_0,\end{aligned}\tag{1.15}$$

where a^R, $\varPhi^R$, $G^{R,2}$ are truncated coefficients. The associated stochastic characteristic system is defined by the following system of Stratonovich's stochastic differential equations

$$\begin{aligned}
\mathrm{d}\varphi_t^0 &= -\frac{1}{4}\partial_\xi G^{R,2}(\varphi_t)\,\mathrm{d}t + \sum_{k=1}^d g_k^R(\varphi_t)\circ \mathrm{d}\beta_k(t),\\
\mathrm{d}\varphi_t^i &= a_i^R(\varphi_t^0)\,\mathrm{d}t, \qquad i=1,\dots,N,
\end{aligned}$$

where the processes φ_t^0 and φ_t^i, $i=1,\dots,N$, respectively, describe the evolution of the ξ-coordinate and x^i-coordinate, $i=1,\dots,N$, respectively, of the characteristic curve. We denote by $\varphi_{s,t}^R(x,\xi)$ its solution starting from (x,ξ) at time s, it defines a stochastic flow of C^3-diffeomorphisms and the corresponding inverse flow is denoted by ψ^R.

If $X_0\in C^{3,\eta}(\mathbb{T}^N\times\mathbb{R})$, $\mathbb{P}$-a.s., then it follows from the Itô-Wentzell formula that

$$X(t,x,\xi;s) = X_0\big(\psi_{s,t}^R(x,\xi)\big)$$

is the unique strong solution to (1.15). We denote by $\mathcal{S}^R$ the solution operator and show that it can be extended to more general function spaces. To be more precise, we set $\mathcal{S}^R(t,s)X_0 = X_0\big(\psi_{s,t}^R(x,\xi)\big)$ and show that $\mathcal{S}^R$ is a family of bounded linear operators on $L^1(\Omega\times\mathbb{T}^N\times\mathbb{R})$ that verifies the semigroup law and for any $X_0\in L^\infty(\Omega\times\mathbb{T}^N\times\mathbb{R})$ there exists a unique weak solution to (1.15) and is represented by $X=\mathcal{S}^R(t,s)X_0$. Clearly, the solutions of (1.15) and (1.13) coincide up to some stopping time. Nevertheless, since the coefficients g_k^R satisfy a linear growth condition independently of R and x, the blow-up cannot occur in a finite time and therefore the pointwise limit

$$\big[\mathcal{S}(t,s)X_0\big](\omega,x,\xi) := \lim_{R\to\infty}\big[\mathcal{S}^R(t,s)X_0\big](\omega,x,\xi), \qquad 0\le s\le t\le T,$$

exists almost surely and $X=\mathcal{S}(t,s)X_0$ is a unique weak solution to (1.13) provided $X_0\in L^\infty(\Omega\times\mathbb{T}^N\times\mathbb{R})$.

1.3.2 Convergence

To investigate the limit of the stochastic BGK model as $\varepsilon\to 0$, we consider the following weak formulation of (1.12) and show its convergence to the kinetic formulation of (1.10). Let $\varphi\in C_c^\infty([0,T)\times\mathbb{T}^N\times\mathbb{R})$ then

$$\begin{aligned}
&\int_0^T \big\langle F^\varepsilon(t),\partial_t\varphi(t)\big\rangle\,\mathrm{d}t + \big\langle F_0^\varepsilon,\varphi(0)\big\rangle + \int_0^T \big\langle F^\varepsilon(t), a\cdot\nabla\varphi(t)\big\rangle\,\mathrm{d}t\\
&= -\frac{1}{\varepsilon}\int_0^T \big\langle \mathbf{1}_{u^\varepsilon(t)>\xi} - F^\varepsilon(t),\varphi(t)\big\rangle\,\mathrm{d}t + \int_0^T \big\langle \partial_\xi F^\varepsilon(t)\,\varPhi\,\mathrm{d}W(t),\varphi(t)\big\rangle \qquad (1.16)\\
&\quad + \frac{1}{2}\int_0^T \big\langle G^2\partial_\xi F^\varepsilon(t),\partial_\xi\varphi(t)\big\rangle\,\mathrm{d}t.
\end{aligned}$$

Remark, that according to the representation formula (1.14), $F^\varepsilon\in[0,1]$, $\varepsilon\in(0,1)$ hence the set of solutions $\{F^\varepsilon;\,\varepsilon\in(0,1)\}$ is bounded in $L^\infty_{\mathcal{P}}(\Omega\times[0,T]\times\mathbb{T}^N\times\mathbb{R})$. Consequently, taking the limit in (1.16) is quite straightforward in all the terms apart from the first one on the right hand side and can be done immediately: by the Banach-Alaoglu theorem, there exists $F\in L^\infty_{\mathcal{P}}(\Omega\times[0,T]\times\mathbb{T}^N\times\mathbb{R})$ such that, up to subsequences,

$$F^\varepsilon \xrightarrow{w^*} F \quad \text{in} \quad L^\infty_{\mathcal{P}}(\Omega\times[0,T]\times\mathbb{T}^N\times\mathbb{R}).$$

Hence, almost surely,

$$\int_0^T \big\langle F^\varepsilon(t),\, \partial_t\varphi(t)\big\rangle\,\mathrm{d}t \longrightarrow \int_0^T \big\langle F(t),\, \partial_t\varphi(t)\big\rangle\,\mathrm{d}t,$$

$$\int_0^T \big\langle F^\varepsilon(t), a\cdot\nabla\varphi(t)\big\rangle\,\mathrm{d}t \longrightarrow \int_0^T \big\langle F(t), a\cdot\nabla\varphi(t)\big\rangle\,\mathrm{d}t,$$

$$\frac{1}{2}\int_0^T \big\langle G^2\partial_\xi F^\varepsilon(t), \partial_\xi\varphi(t)\big\rangle\,\mathrm{d}t \longrightarrow \frac{1}{2}\int_0^T \big\langle G^2\partial_\xi F(t), \partial_\xi\varphi(t)\big\rangle\,\mathrm{d}t.$$

and, according to the hypotheses on the initial data,

$$\big\langle F_0^\varepsilon, \varphi(0)\big\rangle \longrightarrow \big\langle \mathbf{1}_{u_0>\xi}, \varphi(0)\big\rangle.$$

Moreover, by the dominated convergence theorem for stochastic integrals we also deduce the almost sure convergence

$$\int_0^T \big\langle \partial_\xi F^\varepsilon(t)\,\varPhi\,\mathrm{d}W(t), \varphi(t)\big\rangle \longrightarrow \int_0^T \big\langle \partial_\xi F(t)\,\varPhi\,\mathrm{d}W(t), \varphi(t)\big\rangle.$$

In order to obtain the convergence in the remaining term of (1.16) and in view of the kinetic formulation of (1.10), we define

$$m^\varepsilon(\xi) = \frac{1}{\varepsilon}\int_{-\infty}^{\xi} \big(\mathbf{1}_{u^\varepsilon>\zeta} - F^\varepsilon(\zeta)\big)\,\mathrm{d}\zeta$$

which is a nonnegative measure. Due to the convergence in (1.16), for almost every $\omega\in\Omega$ there exists a distribution $m(\omega)$ such that, almost surely,

$$\int_0^T \big\langle m^\varepsilon, \varphi(t)\big\rangle\,\mathrm{d}t \longrightarrow \int_0^T \big\langle m, \varphi(t)\big\rangle\,\mathrm{d}t,$$

for any $\varphi\in C_c^\infty([0,T)\times\mathbb{T}^N\times\mathbb{R})$. Now, it remains to verify that m is a kinetic measure. We start with a uniform estimate for the local densities u^ε. In particular, we get

$$\mathbb{E}\sup_{0\le t\le T}\|u^\varepsilon(t)\|^p_{L^p(\mathbb{T}^N)} \le C$$

which leads to

$$\mathbb{E}\int_{[0,T]\times\mathbb{T}^N\times\mathbb{R}} |\xi|^{2p}\,\mathrm{d}m^\varepsilon(t,x,\xi) \le C.$$

Setting $p=0$, we regard m^ε as random variables with values in $\mathcal{M}_b([0,T]\times\mathbb{T}^N\times\mathbb{R})$[7]. We deduce that the set of laws $\{\mathbb{P}\circ[m^\varepsilon]^{-1};\, \varepsilon\in(0,1)\}$ is tight and therefore any sequence has a weakly convergent subsequence due to the Prokhorov theorem. Consequently, the law of m is supported in $\mathcal{M}_b([0,T]\times\mathbb{T}^N\times\mathbb{R})$ and satisfies also the remaining requirements of the definition of a kinetic measure.

[7] $\mathcal{M}_b([0,T]\times\mathbb{T}^N\times\mathbb{R})$ denotes the space of bounded Borel measures on $[0,T]\times\mathbb{T}^N\times\mathbb{R}$ whose norm is given by the total variation of measures.

1.4 Weak solutions to stochastic differential equations

In Chapter 5, we provide a modified proof of Skorokhod's classical theorem on existence of (weak) solutions to a stochastic differential equation

$$\mathrm{d}X = b(t,X)\,\mathrm{d}t + \sigma(t,X)\,\mathrm{d}W, \quad X(0)=\varphi,$$

where $b : [0,T]\times\mathbb{R}^m \to \mathbb{R}^m$ and $\sigma : [0,T]\times\mathbb{R}^m \to \mathbb{M}_{m\times n}$[8] are Borel functions that are continuous in the second variable. Initially, we assume the linear growth condition which is then relaxed and replaced with a suitable Lyapunov condition. Our proof combines tools that were proposed for handling weak solutions of stochastic evolution equations in infinite-dimensional spaces, where traditional methods cease to work, with results on preservation of the local martingale property under convergence in law. In finite-dimensional situation, the "infinite-dimensional" methods simplify considerably and in our opinion the alternative proof based on them is more lucid and elementary than the standard one and we believe that the reader may find the comparison with other available approaches illuminating.

To explain our argument more precisely, let us recall the structure of the usual proof; for notational simplicity, we shall consider (in the informal introduction only) autonomous equations. Kiyosi Itô showed in his seminal papers (see e.g. [39], [40]) that a stochastic differential equation

$$\mathrm{d}X = b(X)\,\mathrm{d}t + \sigma(X)\,\mathrm{d}W \tag{1.17}$$

$$X(0)=\varphi \tag{1.18}$$

driven by an n-dimensional Wiener process W has a unique solution provided that $b:\mathbb{R}^m\to\mathbb{R}^m$, $\sigma:\mathbb{R}^m\to\mathbb{M}_{m\times n}$ are Lipschitz continuous functions. A next important step was taken by A. Skorokhod ([71], [72]) in 1961, who proved that there exists a solution to (1.17), (1.18) if b and σ are continuous functions of at most linear growth, i.e.

$$\sup_{x\in\mathbb{R}^m}\frac{\|b(x)\|+\|\sigma(x)\|}{1+\|x\|}<\infty.$$

It was realized only later that two different concepts of a solution are involved: for Lipschitzian coefficients, there exists an $(\mathscr{F}_t)$-progressively measurable process in $\mathbb{R}^m$ solving (1.17) and such that $X(0)=\varphi$, whenever $(\Omega,\mathscr{F},(\mathscr{F}_t),\mathbb{P})$ is a stochastic basis carrying an n-dimensional $(\mathscr{F}_t)$-Wiener process and φ is an $\mathscr{F}_0$-measurable function. (We say that (1.17), (1.18) has a strong solution.) On the other hand, for continuous coefficients, a stochastic basis $(\Omega,\mathscr{F},(\mathscr{F}_t),\mathbb{P})$, an n-dimensional $(\mathscr{F}_t)$-Wiener process W and an $(\mathscr{F}_t)$-progressively measurable process X may be found such that X solves (1.17) and $X(0)$ and φ have the same law. (We speak about existence of a weak solution to (1.17), (1.18) in such a case.) It is well known that this difference is substantial in general: under assumptions of the Skorokhod theorem strong solutions need not exist (see [5]).

Skorokhod's existence theorem is remarkable not only by itself, but also because of the method of its proof. To present it, we need some notation: if M and N are continuous real local martingales, then by $\langle M\rangle$ we denote the quadratic variation of M and by $\langle M,N\rangle$ the cross-variation of M and N. Let $M=(M^i)_{i=1}^m$ and $N=(N^j)_{j=1}^n$ be continuous local martingales with values in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. By $\langle\!\langle M\rangle\!\rangle$ we denote

[8] $\mathbb{M}_{m\times n}$ denotes the space of all m-by-n matrices over $\mathbb{R}$ endowed with the Hilbert-Schmidt norm $\|A\|=(\operatorname{Tr}AA^*)^{1/2}$.

the tensor quadratic variation of M, $\langle\!\langle M\rangle\!\rangle = (\langle M^i, M^k\rangle)_{i,k=1}^m$, and we set $\langle M\rangle = \mathrm{Tr}\langle\!\langle M\rangle\!\rangle$. Analogously, we define

$$M \otimes N = \big(M^i N^j\big)_{i=1}^{m}\,{}_{j=1}^{n}, \quad \langle\!\langle M, N\rangle\!\rangle = \big(\langle M^i, N^j\rangle\big)_{i=1}^{m}\,{}_{j=1}^{n}.$$

Let X and Y be random variables with values in the same measurable space $(E, \mathscr{E})$, we write $X \overset{d}{\sim} Y$ if X and Y have the same law on $\mathscr{E}$. Similarly, $X \overset{d}{\sim} \nu$ means that the law of X is a probability measure ν on $\mathscr{E}$.

Let

$$\mathrm{d}X_r = b_r(X_r)\,\mathrm{d}t + \sigma_r(X_r)\,\mathrm{d}W, \quad X_r(0) = \varphi$$

be a sequence of equations which have strong solutions and approximate (1.17) in a suitable sense. (We shall approximate b and σ by Lipschitz continuous functions having the same growth as b and σ, but likewise it is possible to use e.g. finite difference approximations.) The linear growth hypothesis makes it possible to prove that

$$\textit{the laws of } \{X_r;\ r \ge 1\} \textit{ are tight,} \tag{1.19}$$

that is, form a relatively weakly compact set of measures on the space of continuous trajectories. Then Skorokhod's theorem on almost surely converging realizations of converging laws (see e.g. [18], Theorem 11.7.2) may be invoked, which yields a subsequence $\{X_{r_k}\}$ of $\{X_r\}$, a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ and sequences $\{\tilde{X}_k;\ k \ge 0\}$, $\{\tilde{W}_k;\ k \ge 0\}$ such that

$$(X_{r_k}, W) \overset{d}{\sim} (\tilde{X}_k, \tilde{W}_k),\ k \ge 1; \quad (\tilde{X}_k, \tilde{W}_k) \longrightarrow (\tilde{X}_0, \tilde{W}_0),\ \tilde{\mathbb{P}}\text{-a.s.}. \tag{1.20}$$

It is claimed that $\tilde{X}_0$ is the (weak) solution looked for. Skorokhod's papers [71] and [72] are written in a very concise way and details of proofs are not offered; nowadays standard version of Skorokhod's proof is as follows (see [73], Theorem 6.1.6, [37], Theorem IV.2.2, [42], Theorem 5.4.22): under a suitable integrability assumption upon the initial condition,

$$M_k = X_{r_k} - X_{r_k}(0) - \int_0^{\cdot} b_{r_k}(X_{r_k}(s))\,\mathrm{d}s$$

is a martingale with a (tensor) quadratic variation

$$\langle\!\langle M_k\rangle\!\rangle = \int_0^{\cdot} \sigma_{r_k}(X_{r_k}(s))\sigma^*_{r_k}(X_{r_k}(s))\,\mathrm{d}s,$$

for all $k \ge 1$. Equality in law (1.20) implies that also

$$\tilde{M}_k = \tilde{X}_k - \tilde{X}_k(0) - \int_0^{\cdot} b_{r_k}(\tilde{X}_k(s))\,\mathrm{d}s$$

are martingales for $k \ge 1$, with quadratic variations

$$\langle\!\langle \tilde{M}_k\rangle\!\rangle = \int_0^{\cdot} \sigma_{r_k}(\tilde{X}_k(s))\sigma^*_{r_k}(\tilde{X}_k(s))\,\mathrm{d}s.$$

Using convergence $\tilde{\mathbb{P}}$-almost everywhere, it is possible to show that

$$\tilde{M}_0 = \tilde{X}_0 - \tilde{X}_0(0) - \int_0^{\cdot} b(\tilde{X}_0(s))\,\mathrm{d}s$$

is a martingale with a quadratic variation

$$\langle\!\langle \tilde{M}_0 \rangle\!\rangle = \int_0^{\cdot} \sigma(\tilde{X}_0(s))\sigma^*(\tilde{X}_0(s))\,\mathrm{d}s.$$

By the integral representation theorem for martingales with an absolutely continuous quadratic variation (see e.g. [42], Theorem 3.4.2, or [8], Theorem II.7.1'), there exists a Wiener process $\hat{W}$ (on an extended probability space) satisfying

$$\tilde{M}_0 = \int_0^{\cdot} \sigma(\tilde{X}_0(s))\,\mathrm{d}\hat{W}(s).$$

Therefore, $(\hat{W}, \tilde{X}_0)$ is a weak solution to (1.17), (1.18). (In the cited books, martingale problems are used instead of weak solutions. Then the integral representation theorem is hidden in the construction of a weak solution from a solution to the martingale problem, so a complete proof is essentially the one sketched above.)

This procedure has two rather nontrivial inputs: the Skorokhod representation theorem, and the integral representation theorem whose proof, albeit based on a simple and beautiful idea, becomes quite technical if the space dimension is greater than one. An alternative approach to identification of the limit was discovered recently (see [11], [60]) in the course of study of stochastic wave maps between manifolds, where integral representation theorems for martingales are no longer available. The new method, which refers only to basic properties of martingales and stochastic integrals, may be described in the case of the problem (1.17), (1.18) in the following way: One starts again with a sequence $\{(\tilde{X}_k, \tilde{W}_k)\}$ such that (1.20) holds true. If the initial condition is p-integrable for some $p > 2$, it can be shown in a straightforward manner, using the almost sure convergence, that

$$\tilde{M}_0, \quad \|\tilde{M}_0\|^2 - \int_0^{\cdot} \|\sigma(\tilde{X}_0(s))\|^2\,\mathrm{d}s, \quad \tilde{M}_0 \otimes \tilde{W}_0 - \int_0^{\cdot} \sigma(\tilde{X}_0(s))\,\mathrm{d}s$$

are martingales, in other words,

$$\left\langle \tilde{M}_0 - \int_0^{\cdot} \sigma(\tilde{X}_0(s))\,\mathrm{d}\tilde{W}_0(s) \right\rangle = 0,$$

whence one concludes that $(\tilde{W}_0, \tilde{X}_0)$ is a weak solution. If the additional integrability hypothesis on φ is not satisfied, the proof remains almost the same, only a suitable cut-off procedure must be amended.

We take a step further and eliminate also the Skorokhod representation theorem. Let $\tilde{\mathbb{P}}_k$ be the laws of (X_{r_k}, W) on the space $U = \mathscr{C}([0,T];\mathbb{R}^m) \times \mathscr{C}([0,T];\mathbb{R}^n)$; we know that the sequence $\{\tilde{\mathbb{P}}_k\}$ converges weakly to some measure $\tilde{\mathbb{P}}_0$. Denote by (Y, B) the canonical process on U and set

$$\bar{M}_k = Y - Y(0) - \int_0^{\cdot} b_{r_k}(Y(s))\,\mathrm{d}s, \quad k \geq 0$$

(with $b_{r_0} = b$, $\sigma_{r_0} = \sigma$). Then

$$\bar{M}_k, \quad \|\bar{M}_k\|^2 - \int_0^{\cdot} \|\sigma_{r_k}(Y(s))\|^2\,\mathrm{d}s, \quad \bar{M}_k \otimes B - \int_0^{\cdot} \sigma_{r_k}(Y(s))\,\mathrm{d}s, \tag{1.21}$$

are local martingales under the measure $\tilde{\mathbb{P}}_k$ for every $k \geq 1$, as can be inferred quite easily from the definition of the measure $\tilde{\mathbb{P}}_k$. Now one may try to use Theorem IX.1.17 from [41] stating, roughly speaking, that a limit in law of a sequence of continuous local martingales is a local martingale. We do not use this theorem explicitly, since to establish convergence in law of the processes (1.21) as $k \to \infty$ is not simpler than to check the local martingale property for $k = 0$ directly, but our argument is inspired by the proofs in the book [41]. The proof we propose is not difficult and it is almost self-contained, it requires only two auxiliary lemmas (with simple proofs) from [41] on continuity properties of certain first entrance times which we recall in Appendix. Once we know that the processes (1.21) are local martingales for $k = 0$ as well, the trick from [11] and [60] may be used yielding that (B, Y) is a weak solution to (1.17), (1.18). It is worth mentioning that this procedure is independent of any integrability hypothesis on φ.

The proof of (1.19) not being our main concern notwithstanding, we decided to include a less standard proof of tightness inspired also by the theory of stochastic partial differential equations. We adopt an argument proposed by D. Gątarek and B. Gołdys in [27] (cf. also [15], Chapter 8), who introduced it when studying weak solutions to stochastic evolution equations in Hilbert spaces, and which relies on the factorization method of G. Da Prato, S. Kwapień and J. Zabczyk (see [15], Chapters 5 and 7, for a thorough exposition) and on compactness properties of fractional integral operators. The fractional calculus has become popular amongst probabilists recently because of its applications to fractional Brownian motion driven stochastic integrals and a proof of tightness using it may suit some readers more than the traditional one based on estimates of moduli of continuity.

The precise result to be proved by this method reads as follows.

Theorem 1.4.1. *Let $b : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ and $\sigma : [0,T] \times \mathbb{R}^m \to \mathbb{M}_{m\times n}$ be Borel functions such that $b(t,\cdot)$ and $\sigma(t,\cdot)$ are continuous on $\mathbb{R}^m$ for any $t \in [0,T]$ and the linear growth hypothesis is satisfied, that is*

$$\exists K_* < \infty \ \forall t \in [0,T] \ \forall x \in \mathbb{R}^m \quad \|b(t,x)\| \vee \|\sigma(t,x)\| \le K_*\big(1 + \|x\|\big).$$

Let ν be a Borel probability measure on $\mathbb{R}^m$. Then there exists a weak solution to the problem

$$\mathrm{d}X = b(t,X)\,\mathrm{d}t + \sigma(t,X)\,\mathrm{d}W, \qquad X(0) \overset{d}{\sim} \nu. \tag{1.22}$$

Furthermore, it turns out that this new method can be used even if the linear growth condition is relaxed to existence of a suitable Lyapunov function. Namely, we proved the following result.

Theorem 1.4.2. *Assume that a hypothesis*

(A) *$b(r,\cdot)$ and $\sigma(r,\cdot)$ are continuous on $\mathbb{R}^m$ for any $r \in [0,T]$ and both functions b, σ are locally bounded on $[0,T] \times \mathbb{R}^m$, i.e.*

$$\sup_{r\in[0,T]} \sup_{\|z\|\le L} \big\{\|b(r,z)\| \vee \|\sigma(r,z)\|\big\} < \infty$$

for all $L \ge 0$,

is satisfied and a function $V \in \mathscr{C}^2(\mathbb{R}^m)$ may be found such that

(L1) *there exists an increasing function $\kappa\colon \mathbb{R}_+ \to]0,\infty[$ such that*

$$\lim_{r\to\infty} \kappa(r) = +\infty$$

and $V(x) \geq \kappa(\|x\|)$ *for all* $x \in \mathbb{R}^m$,

(L2) *there exists* $\gamma \geq 0$ *such that*

$$\big\langle b(t,x), DV(x)\big\rangle + \frac{1}{2}\operatorname{Tr}\big(\sigma(t,x)^* D^2V(x)\sigma(t,x)\big) \leq \gamma V(x)$$

for all $(t,x) \in [0,T] \times \mathbb{R}^m$.

Then there exists a weak solution to (1.22).

Chapter 2

Strong Solutions of Semilinear Stochastic Partial Differential Equations

Abstract: We study the Cauchy problem for a semilinear stochastic partial differential equation driven by a finite-dimensional Wiener process. In particular, under the hypothesis that all the coefficients are sufficiently smooth and have bounded derivatives, we consider the equation in the context of power scale generated by a strongly elliptic differential operator. Application of semigroup arguments then yields the existence of a continuous strong solution.

Results of this chapter were published under the title:

- M. Hofmanová, *Strong Solutions of Semilinear Stochastic Partial Differential Equations*, Nonlinear Differ. Equ. Appl. **20** (3) (2013) 757–778.

2.1 Introduction

In the present paper, we consider the following semilinear stochastic partial differential equation driven by a finite-dimensional Wiener process:

$$\begin{aligned} \mathrm{d}u &= \big[\mathcal{A}u + F(u)\big]\mathrm{d}t + \sigma(u)\,\mathrm{d}W, \quad x\in\mathbb{T}^N,\ t\in(0,T),\\ u(0) &= u_0, \end{aligned} \tag{2.1}$$

where $-\mathcal{A}$ is a strongly elliptic differential operator, F is generally nonlinear unbounded operator and the diffusion coefficient in the stochastic term is also nonlinear.

It is a well known fact in the field of PDEs and SPDEs that many equations do not, in general, have classical or strong solutions and can be solved only in some weaker sense. Unlike deterministic problems, in the case of stochastic equations we can only ask whether the solution is smooth in the space variable. Thus, the aim of the present work is to determine conditions on coefficients and initial data under which there exists a spatially smooth solution to (2.1). The motivation for such a regularity result came from our research in the field of degenerate parabolic SPDEs of second order (see [32]), where smooth solutions of certain approximate nondegenerate problems were needed in order to derive the so-called kinetic formulation and to obtain kinetic solution. Nevertheless, since the regularity result of the present paper is based on properties of strongly elliptic operators, generalization to higher order equations does not cause any additional problems.

The literature devoted to the existence of a classical solution to deterministic parabolic problems is quite extensive, let us mention for instance the works of Friedman [25], Grunau, von Wahl [28], Ladyzhenskaya, Solonnikov, Ural'ceva [51], Lieberman [53], von Wahl [77], Yagi [79] and the references therein. Regularity in the case of linear parabolic SPDEs was treated by Krylov [46], Krylov and Rozovskii [47], [48] and the references therein, and Flandoli [21]. However, there seems to be less papers concentrated on regularity for nonlinear SPDEs. The starting point for our research was the paper of Gyöngy and Rovira [30] who studied a class of second order parabolic semilinear SPDEs. However, they were only concerned with L^p-valued solutions so our work can be regarded as an extension of their result. Related problems were also discussed by Zhang [80], [81], nevertheless, his assumptions are not satisfied in our case.

The main difficulty in the case of semilinear equations lies in the nonlinearities F and σ as, in higher order Sobolev spaces, we cannot expect the Lipschitz condition to be satisfied and hence the fixed point argument cannot be applied. This issue is closely related to the mapping properties of Nemytskij operators, i.e. $T_G : h \mapsto G(h)$, where h belongs to some function space E and $G:\mathbb{R}\to\mathbb{R}$ is nonlinear. It turns out (and was discussed in-depth in the book of Runst and Sickel [68]) that the mapping properties of these operators depend strongly on the chosen domain of definition and even for E being a Sobolev space they do not, in general, map E to itself.

Let us make things clearer on a simple example of a heat equation with a nonlinear right-hand side

$$\partial_t u = \Delta u + H(u), \qquad x\in\mathbb{T}^N,\ t\in(0,T). \tag{2.2}$$

Let $p\in[1,\infty)$. If $H:\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous then

$$\|H(z_1)-H(z_2)\|_{L^p(\mathbb{T}^N)} \le C\|z_1-z_2\|_{L^p(\mathbb{T}^N)}, \quad z_1,\,z_2\in L^p(\mathbb{T}^N),$$

therefore, as an easy consequence of the Banach fixed point theorem, there exists a unique mild solution to (2.2) in $L^p(\mathbb{T}^N)$. However, if $m \geq 1$ it is not generally true that

$$\|H(z_1) - H(z_2)\|_{W^{m,p}(\mathbb{T}^N)} \leq C\|z_1 - z_2\|_{W^{m,p}(\mathbb{T}^N)}, \quad z_1,\, z_2 \in W^{m,p}(\mathbb{T}^N),$$

so the existence of a solution in higher order Sobolev spaces cannot be proved directly. In fact, even the linear growth condition fails for $m \geq 2$ since the norm of a superposition does not grow linearly with the norm of the inner function. For example, if we consider $2 \leq m \leq N/p$, $p \in (1, \infty)$, then only linear operators map $W^{m,p}(\mathbb{T}^N)$ to itself (see [68, Theorem 5.2.4/2]).

On the other hand, for any $m \in \mathbb{N}$ and $p \in [1, \infty)$, under the hypothesis of a sufficiently smooth function H having bounded derivatives one arrives at the fact that the Nemytskij operator T_H maps $W^{1,mp}(\mathbb{T}^N) \cap W^{m,p}(\mathbb{T}^N)$ to itself and the following estimate holds true for any $z \in W^{m,p}(\mathbb{T}^N) \cap W^{1,mp}(\mathbb{T}^N)$ (cf. Proposition 2.3.1, Corollary 2.3.2 and Remark 2.3.3)

$$\|H(z)\|_{W^{m,p}(\mathbb{T}^N)} \leq C\big(1 + \|z\|_{W^{m,p}(\mathbb{T}^N)} + \|z\|^m_{W^{1,mp}(\mathbb{T}^N)}\big).$$

It turns out to be the keystone of our proof of regularity. In particular, we proceed successively in several steps. First of all, we consider the equation (2.2) in $L^{mp}(\mathbb{T}^N)$ and apply the Banach fixed point theorem to conclude the existence of an $L^{mp}(\mathbb{T}^N)$-valued mild solution. Next, we study its Picard iterations as processes having values in the Sobolev spaces $W^{1,mp}(\mathbb{T}^N)$. Having known that T_H maps $W^{1,mp}(\mathbb{T}^N)$ to itself we are able to find a uniform estimate of the $W^{1,mp}(\mathbb{T}^N)$-norm which is then used in the last step to deduce a uniform estimate of the $W^{m,p}(\mathbb{T}^N)$-norm. Both estimates remain valid also for the limit process and, as a consequence, the mild solution to (2.2) is even strong (for a detailed exposition of these two concepts of solution we refer the reader to [15]).

Unlike the introduction, in the proof of the main result, Theorem 2.2.1, the integrability exponent p is only allowed to take values in $[2, \infty)$ which is given by the use of the stochastic Itô integration in 2-smooth Banach spaces (see [10], [61]).

As an immediate consequence of the main result, we obtain a continuous $C^{k,\lambda}$-valued solution. Here, we use the Sobolev embedding theorem so the stochastic integration in Banach spaces, i.e. $W^{m,p}$, allows us to weaken the smoothness assumptions on coefficients. We note that the regularity of the solution depends only on the regularity of the coefficients and the initial data and is not limited by the order of the equation.

The paper is organized as follows. In Section 2, we review the basic setting and state our main result. In Section 3, we collect important preliminary results related to Nemytskij operators. In the final section, these results are applied and the proof of the main theorem is established.

2.2 Setting and the main result

Let us first introduce the notation which will be used later on. We will consider periodic boundary conditions: $x \in \mathbb{T}^N$ where $\mathbb{T}^N$ is the N-dimensional torus. The Sobolev spaces on $\mathbb{T}^N$ will be denoted by $W^{m,p}(\mathbb{T}^N)$ and by $W^{m,p}(\mathbb{T}^N;\mathbb{R}^n)$ we will denote the space of all functions $z = (z_1, \dots, z_n) : \mathbb{T}^N \to \mathbb{R}^n$ such that $z_i \in W^{m,p}(\mathbb{T}^N)$, $i = 1, \dots, n$.

We now give the precise assumptions on each of the terms appearing in the above equation (2.1). We will work on a finite-time interval $[0, T]$, $T > 0$. The operator $-\mathcal{A}$ is a strongly elliptic differential operator of order $2l$ with variable coefficients of class $C^\infty(\mathbb{T}^N)$. Let us assume, in addition, that $-\mathcal{A}$ is formally symmetric and positive, i.e.

we assume that 0 belongs to the resolvent set of $-\mathcal{A}$. As an example of this operator let us mention for instance the second order differential operator in divergence form given by

$$\mathcal{A}u=\sum_{i,j=1}^{N}\partial_{x_i}\big(A_{ij}(x)\partial_{x_j}u\big),$$

where the coefficients $A_{ij}=A_{ji}$ are real-valued smooth functions and satisfy the uniform ellipticity condition, i.e. there exists $\alpha>0$ such that

$$\sum_{i,j=1}^{N}A_{ij}(x)\zeta_i\zeta_j\geq\alpha|\zeta|^2,\qquad\forall x\in\mathbb{T}^N,\ \ \forall\zeta\in\mathbb{R}^N.$$

Let us now collect basic facts concerning strongly elliptic differential operators satisfying our hypotheses (for a detailed exposition we refer the reader to [62]). Set $D(\mathcal{A}_p)=W^{2l,p}(\mathbb{T}^N)$. Then the linear unbounded operator $\mathcal{A}_p$ in $L^p(\mathbb{T}^N)$ defined by

$$\mathcal{A}_pu=\mathcal{A}u,\qquad u\in D(\mathcal{A}_p),$$

is the infinitesimal generator of a bounded analytic semigroup on $L^p(\mathbb{T}^N)$. Let us denote this semigroup by $\mathcal{S}_p$. Fractional powers of $-\mathcal{A}_p$ are well defined and their domains correspond to classical Sobolev spaces (see [1, Section 10]), i.e.

$$\Big(D\big((-\mathcal{A}_p)^\delta\big),\big\|(-\mathcal{A}_p)^\delta\cdot\big\|_{L^p(\mathbb{T}^N)}\Big)\cong\big(W^{2l\delta,p}(\mathbb{T}^N),\|\cdot\|_{W^{2l\delta,p}(\mathbb{T}^N)}\big),\qquad\delta\geq0.$$

We will also make use of the following property of analytic semigroups (see [62, Chapter 2, Theorem 6.13]):

$$\begin{aligned}\forall t>0\quad\forall\delta>0\quad\text{the operator }\ (-\mathcal{A}_p)^\delta\mathcal{S}_p(t)\ \text{ is bounded in }L^p(\mathbb{T}^N),\\ \|(-\mathcal{A}_p)^\delta\mathcal{S}_p(t)\|\leq C_{\delta,p}\,t^{-\delta}\end{aligned}\tag{2.3}$$

(here $\|\cdot\|$ stands for the operator norm).

The nonlinear term F is defined as follows: for any $p\in[2,\infty)$

$$\begin{aligned}F:L^p(\mathbb{T}^N)&\longrightarrow W^{-2l+1,p}(\mathbb{T}^N)\\ z&\longmapsto\sum_{|\alpha|\leq 2l-1}a_\alpha\,\mathrm{D}^\alpha f_\alpha(z),\end{aligned}$$

where $a_\alpha\in\mathbb{R}$ and the functions f_α, $|\alpha|\leq 2l-1$, are smooth enough (exact assumptions will be given later). Let us denote by f the vector of functions $(f_\alpha\,;|\alpha|\leq 2l-1,\ a_\alpha\neq0)$ and denote its length by η.

Throughout this article we fix a stochastic basis $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq0},\mathbb{P})$ with a complete, right-continuous filtration. Let $\mathcal{P}$ denote the predictable σ-algebra on $\Omega\times[0,T]$ associated with $(\mathscr{F}_t)_{t\geq0}$. For simplicity we will only consider finite-dimensional noise, however, the result can be extended to the infinite-dimensional case. Let $\mathfrak{U}$ be a finite-dimensional Hilbert space and let $\{e_i\}_{i=1}^d$ be its orthonormal basis. The process W is a d-dimensional $(\mathscr{F}_t)$-Wiener process in $\mathfrak{U}$, i.e. it has an expansion of the form $W(t)=\sum_{i=1}^dW_i(t)\,e_i$, where W_i, $i=1,\dots,d$, are mutually independent real-valued standard Wiener processes relative to $(\mathscr{F}_t)_{t\geq 0}$. The diffusion coefficient σ is then defined as

$$\begin{aligned}\sigma(z) : \mathfrak{U} &\longrightarrow L^p(\mathbb{T}^N)\\ h &\longmapsto \sum_{i=1}^{d} \sigma_i(\cdot, z(\cdot))\langle e_i, h\rangle, \qquad z \in L^p(\mathbb{T}^N),\end{aligned}$$

where the functions $\sigma_1, \dots, \sigma_d : \mathbb{T}^N \times \mathbb{R} \to \mathbb{R}$ satisfy the following linear growth condition

$$\sum_{i=1}^{d} \big|\sigma_i(x,\xi)\big|^2 \leq C\big(1+|\xi|^2\big), \qquad x\in\mathbb{T}^N,\ \xi\in\mathbb{R}. \tag{2.4}$$

Since we are going to solve (2.1) in $L^p(\mathbb{T}^N)$, for $p\in[2,\infty)$, we need to ensure the existence of the stochastic integral as an $L^p(\mathbb{T}^N)$-valued process. Recall, that L^p spaces, $p\in[2,\infty)$, as well as the Sobolev spaces $W^{m,p}$, $p\in[2,\infty)$, $m\geq 0$, belong to a class of the so-called 2-smooth Banach spaces, which are well suited for stochastic Itô integration. (A detailed construction of stochastic integral for processes with values in 2-smooth Banach spaces can be found in [10] or [61].) Let us denote by $\gamma(\mathfrak{U};X)$ the space of all γ-radonifying operators from $\mathfrak{U}$ to a 2-smooth Banach space X. We will show that $\sigma(z)\in\gamma(\mathfrak{U};L^p(\mathbb{T}^N))$ for any $z\in L^p(\mathbb{T}^N)$ and

$$\|\sigma(z)\|^2_{\gamma(\mathfrak{U};L^p(\mathbb{T}^N))} \leq C\big(1+\|z\|^2_{L^p(\mathbb{T}^N)}\big).$$

Note, that the following fact holds true:

$$\begin{aligned}&\forall s>0 \quad \exists C_s\in(0,\infty) \quad \forall\gamma_1,\dots,\gamma_d \text{ independent } \mathcal{N}(0,1)\text{-random variables}\\ &\qquad \forall r_1,\dots,r_d\in\mathbb{R} \qquad \bigg(\mathbb{E}\Big|\sum_{i=1}^{d} r_i\gamma_i\Big|^s\bigg)^{\frac{1}{s}} = C_s\bigg(\sum_{i=1}^{d} r_i^2\bigg)^{\frac12}.\end{aligned} \tag{2.5}$$

The proof is, by the way, easy: $\big(\sum_{i=1}^d r_i^2\big)^{-\frac12}\sum_{i=1}^d r_i\gamma_i$ is an $\mathcal{N}(0,1)$-random variable. Let $\{\gamma_i\}_{i=1}^d$ be a sequence of independent $\mathcal{N}(0,1)$-random variables, by the definition of a γ-radonifying norm, using (2.5) and (2.4)

$$\begin{aligned}\|\sigma(z)\|^2_{\gamma(\mathfrak{U};L^p(\mathbb{T}^N))} &= \mathbb{E}\Big\|\sum_{i=1}^{d}\gamma_i\,\sigma(z)e_i\Big\|^2_{L^p(\mathbb{T}^N)} = \mathbb{E}\Big\|\sum_{i=1}^{d}\gamma_i\,\sigma_i(\cdot,z(\cdot))\Big\|^2_{L^p(\mathbb{T}^N)}\\ &\leq \bigg(\mathbb{E}\Big\|\sum_{i=1}^{d}\gamma_i\,\sigma_i(\cdot,z(\cdot))\Big\|^p_{L^p(\mathbb{T}^N)}\bigg)^{\frac2p} = \bigg(\int_{\mathbb{T}^N}\mathbb{E}\Big|\sum_{i=1}^{d}\gamma_i\,\sigma_i(y,z(y))\Big|^p\mathrm{d}y\bigg)^{\frac2p}\\ &= C_p^2\bigg(\int_{\mathbb{T}^N}\Big(\sum_{i=1}^{d}\big|\sigma_i(y,z(y))\big|^2\Big)^{\frac p2}\mathrm{d}y\bigg)^{\frac2p} \leq C\bigg(\int_{\mathbb{T}^N}\big(1+|z(y)|^2\big)^{\frac p2}\mathrm{d}y\bigg)^{\frac2p}\\ &\leq C\big(1+\|z\|^2_{L^p(\mathbb{T}^N)}\big)\end{aligned} \tag{2.6}$$

and the claim follows. In this paper, the letter C denotes a positive constant, which is unimportant and may change from one line to another.

Let us close this section by stating the main result to be proved precisely.

Theorem 2.2.1. *Let $p\in[2,\infty)$, $q\in(2,\infty)$, $m\in\mathbb{N}$. We suppose that*

$$u_0\in L^q(\Omega;W^{m,p}(\mathbb{T}^N))\cap L^{mq}(\Omega;W^{1,mp}(\mathbb{T}^N))$$

and

$$f_\alpha \in C^m(\mathbb{R}) \cap C^{2l-1}(\mathbb{R}),\ \ |\alpha| \le 2l-1; \qquad \sigma_i \in C^m(\mathbb{T}^N \times \mathbb{R}),\ \ i = 1, \dots, d,$$

have bounded derivatives up to order m. Then there exists a unique solution to (2.1) which belongs to

$$L^q(\Omega; C([0,T]; W^{m,p}(\mathbb{T}^N))) \cap L^{mq}(\Omega; C([0,T]; W^{1,mp}(\mathbb{T}^N)))$$

and the following estimate holds true

$$\begin{aligned} \mathbb{E} \sup_{0 \le t \le T} \|u(t)\|^q_{W^{m,p}(\mathbb{T}^N)} + \mathbb{E} \sup_{0 \le t \le T} \|u(t)\|^{mq}_{W^{1,mp}(\mathbb{T}^N)} \\ \le C\big(1 + \mathbb{E}\|u_0\|^q_{W^{m,p}(\mathbb{T}^N)} + \mathbb{E}\|u_0\|^{mq}_{W^{1,mp}(\mathbb{T}^N)}\big). \end{aligned}$$

Corollary 2.2.2. *Let $k \in \mathbb{N}_0$ and $u_0 \in L^q(\Omega; C^{k+1}(\mathbb{T}^N))$ for all $q \in (2, \infty)$. Assume that*

$$f_\alpha \in C^{k+1}(\mathbb{R}) \cap C^{2l-1}(\mathbb{R}),\ |\alpha| \le 2l-1; \qquad \sigma_i \in C^{k+1}(\mathbb{T}^N \times \mathbb{R}),\ i = 1, \dots, d,$$

have bounded derivatives up to order $k+1$. Then there exists a solution to (2.1) which belongs to

$$L^q(\Omega; C([0,T]; C^{k,\lambda}(\mathbb{T}^N))) \qquad \textit{for every} \quad \lambda \in (0,1).$$

Remark 2.2.3. In the proof, we show regularity of the mild solution, however, the resulting estimates imply that it is even strong (see [15] for a thorough exposition of these two concepts of solution).

2.3 Preliminaries

For the reader's convenience we shall first restate the following auxiliary result which is taken from [68, Theorem 5.2.5].

Proposition 2.3.1. *Let $m \in \mathbb{N}$, $m \ge 2$, $p \in [1, \infty)$. Suppose that the function $G \in C^m(\mathbb{R})$ has bounded derivatives up to order m. If $h \in W^{m,p}(\mathbb{T}^N) \cap W^{1,mp}(\mathbb{T}^N)$ then the following estimate holds true*

$$\big\|G(h)\big\|_{W^{m,p}(\mathbb{T}^N)} \le C\big(1 + \|h\|^m_{W^{1,mp}(\mathbb{T}^N)} + \|h\|_{W^{m,p}(\mathbb{T}^N)}\big)$$

with a constant independent of h.

Proof. Since G has a linear growth we have

$$\|G(h)\|_{L^p(\mathbb{T}^N)} \le C\big(1 + \|h\|_{L^p(\mathbb{T}^N)}\big).$$

Next, we will employ the chain rule formula for partial derivatives of compositions:

$$\mathrm{D}^\beta G(h(x)) = \sum_{l=1}^{|\beta|} \sum_{\substack{\alpha_1 + \dots + \alpha_l = \beta \\ |\alpha_i| \ne 0}} C_{\beta,l,\alpha_1,\dots,\alpha_l}\, G^{(l)}(h(x))\, \mathrm{D}^{\alpha_1} h(x) \cdots \mathrm{D}^{\alpha_l} h(x),$$

where $\beta = (\beta_1, \dots, \beta_N)$, $\alpha_i = (\alpha_i^1, \dots, \alpha_i^N)$, $i = 1, \dots, l$, are multiindices and $C_{\beta,l,\alpha_1,\dots,\alpha_l}$ are certain combinatorial constants. It is sufficient to consider $|\beta| = m$. By the Hölder inequality we obtain

$$\big\|G^{(l)}(h)\,\mathrm{D}^{\alpha_1}h\cdots\mathrm{D}^{\alpha_l}h\big\|_{L^p(\mathbb{T}^N)}\leq\big\|G^{(l)}\big\|_{L^\infty(\mathbb{R})}\prod_{i=1}^{l}\big\|\mathrm{D}^{\alpha_i}h\big\|_{L^{\frac{mp}{|\alpha_i|}}(\mathbb{T}^N)}.$$

Due to interpolation inequalities, we have

$$\|h\|_{W^{|\alpha_i|,\frac{mp}{|\alpha_i|}}(\mathbb{T}^N)}\leq C\|h\|^{1-\theta_i}_{W^{1,mp}(\mathbb{T}^N)}\|h\|^{\theta_i}_{W^{m,p}(\mathbb{T}^N)}\qquad\text{with}\qquad\theta_i=\frac{|\alpha_i|-1}{m-1}.$$

Therefore

$$\begin{aligned}\big\|\mathrm{D}^\beta G(h)\big\|_{L^p(\mathbb{T}^N)}&\leq C\max_{1\leq l\leq m}\sum_{\substack{\alpha_1+\cdots+\alpha_l=\beta\\|\alpha_i|\neq0}}\prod_{i=1}^{l}\|h\|^{1-\theta_i}_{W^{1,mp}(\mathbb{T}^N)}\,\|h\|^{\theta_i}_{W^{m,p}(\mathbb{T}^N)}\\&\leq C\max_{1\leq l\leq m}\|h\|^{l-\frac{m-l}{m-1}}_{W^{1,mp}(\mathbb{T}^N)}\|h\|^{\frac{m-l}{m-1}}_{W^{m,p}(\mathbb{T}^N)}\\&\leq C\big(\|h\|^m_{W^{1,mp}(\mathbb{T}^N)}+\|h\|_{W^{m,p}(\mathbb{T}^N)}\big),\end{aligned}$$

where we used the fact that the function $y\mapsto a^y(b/a)^{\frac{m-y}{m-1}}$ is monotone so the maximal value is attained at $y=1$ or $y=m$. The proof is complete. □

This result can be easily extended to more general outer function.

Corollary 2.3.2. *Let $m\in\mathbb{N}, m\geq2, p\in[1,\infty)$. Suppose that the function $G\in C^m(\mathbb{T}^N\times\mathbb{R})$ has bounded derivatives up to order m. If $h\in W^{m,p}(\mathbb{T}^N)\cap W^{1,mp}(\mathbb{T}^N)$ then the following estimate holds true*

$$\big\|G(\cdot,h(\cdot))\big\|_{W^{m,p}(\mathbb{T}^N)}\leq C\big(1+\|h\|^m_{W^{1,mp}(\mathbb{T}^N)}+\|h\|_{W^{m,p}(\mathbb{T}^N)}\big)$$

with a constant independent of h.

Remark 2.3.3. The situation is much easier for the first order derivatives: fix $p\in[1,\infty)$ and let $h\in W^{1,p}(\mathbb{T}^N)$

(i) if $G\in C^1(\mathbb{R})$ has a bounded derivative then

$$\big\|G(h)\|_{W^{1,p}(\mathbb{T}^N)}\leq C\big(1+\|h\|_{W^{1,p}(\mathbb{T}^N)}\big),$$

(ii) if $G\in C^1(\mathbb{T}^N\times\mathbb{R})$ has bounded derivatives then

$$\big\|G(\cdot,h(\cdot))\|_{W^{1,p}(\mathbb{T}^N)}\leq C\big(1+\|h\|_{W^{1,p}(\mathbb{T}^N)}\big),$$

where the constant C is independent of h.

2.4 Proof of the main result

Let us review the main ideas of the proof. The proof is divided into three steps. In the first step, we apply the Banach fixed point theorem to conclude the existence of an $L^{mp}(\mathbb{T}^N)$-valued mild solution of (2.1). In the second step, we study Picard iterations for (2.1) and find a uniform estimate of the $W^{1,mp}(\mathbb{T}^N)$-norm. It is then used in the third step to derive a uniform estimate of the $W^{m,p}(\mathbb{T}^N)$-norm. This estimate remains valid also for the limit process and the statement follows.

These steps will be stated as propositions.

Proposition 2.4.1 (Fixed point argument)**.** *Let $p, q \in [2,\infty)$. Assume that $u_0 \in L^q(\Omega; L^p(\mathbb{T}^N))$ and*

$$f_\alpha \in C^{2l-1}(\mathbb{R}),\ \ |\alpha| \leq 2l-1; \qquad\qquad \sigma_i \in C^1(\mathbb{T}^N \times \mathbb{R}),\ \ i = 1,\dots,d,$$

have bounded derivatives of first order. Then there exists a unique mild solution to (2.1) *which belongs to*

$$L^q(\Omega \times [0,T], \mathcal{P}, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t; L^p(\mathbb{T}^N)).$$

Proof. Let us denote

$$\mathscr{H} = L^q(\Omega \times [0,T], \mathcal{P}, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t; L^p(\mathbb{T}^N))$$

and define the mapping

$$\begin{aligned}\big(\mathscr{K} v\big)(t) &= \mathcal{S}_p(t)u_0 + \int_0^t \mathcal{S}_p(t-s)F(v(s))\,\mathrm{d}s + \int_0^t \mathcal{S}_p(t-s)\sigma(v(s))\,\mathrm{d}W(s)\\ &= \mathcal{S}_p(t)u_0 + \big(\mathscr{K}_1 v\big)(t) + \big(\mathscr{K}_2 v\big)(t), \qquad t \in [0,T],\ v \in \mathscr{H}.\end{aligned}$$

Here, we employ stochastic integration in $L^p(\mathbb{T}^N)$ as introduced in Section 2.2. We shall prove that $\mathscr{K}$ maps $\mathscr{H}$ into $\mathscr{H}$ and that it is a contraction.

Since $u_0 \in L^q(\Omega; L^p(\mathbb{T}^N))$ it follows easily that $\mathcal{S}_p(t)u_0 \in \mathscr{H}$. In order to estimate the second term, let $\delta = \frac{2l-1}{2l}$ and note that

$$\mathcal{S}_p(t-s)F(v(s)) = \mathcal{S}_p(t-s)(-\mathcal{A}_p)^{\delta}(-\mathcal{A}_p)^{-\delta} \sum_{\substack{|\alpha| \leq 2l-1\\ a_\alpha \neq 0}} a_\alpha \mathrm{D}^\alpha f_\alpha(v(s)),$$

where the operator $(-\mathcal{A}_p)^{\delta}$ commutes with the semigroup and the operator

$$\begin{aligned}\mathcal{B}_p\ :\ L^p(\mathbb{T}^N;\mathbb{R}^\eta) &\longrightarrow L^p(\mathbb{T}^N)\\ \{z_\alpha\}_{\substack{|\alpha| \leq 2l-1\\ a_\alpha \neq 0}} &\longmapsto (-\mathcal{A}_p)^{-\delta} \sum_{\substack{|\alpha| \leq 2l-1\\ a_\alpha \neq 0}} a_\alpha \mathrm{D}^\alpha z_\alpha\end{aligned}$$

is bounded. Indeed, let p^* be the conjugate exponent to p. Then the operator $L^{p^*}(\mathbb{T}^N) \to L^{p^*}(\mathbb{T}^N),\ v \mapsto a_\alpha \mathrm{D}^\alpha (-\mathcal{A}_{p^*})^{-\delta} v$, $|\alpha| \leq 2l-1$, is clearly bounded so for $z \in L^p(\mathbb{T}^N;\mathbb{R}^\eta)$

we have

$$
\begin{aligned}
&\Big\|(-\mathcal{A}_p)^{-\delta}\sum_{\substack{|\alpha|\le 2l-1\\ a_\alpha\neq 0}} a_\alpha \mathrm{D}^\alpha z_\alpha\Big\|_{L^p(\mathbb{T}^N)}\\
&\quad= \sup_{\substack{v\in L^{p^*}(\mathbb{T}^N)\\ \|v\|_{L^{p^*}(\mathbb{T}^N)}\le 1}} \left|\int_{\mathbb{T}^N}(-\mathcal{A}_p)^{-\delta}\sum_{\substack{|\alpha|\le 2l-1\\ a_\alpha\neq 0}} a_\alpha \mathrm{D}^\alpha z_\alpha(x)\, v(x)\,\mathrm{d}x\right|\\
&\quad= \sup_{\substack{v\in L^{p^*}(\mathbb{T}^N)\\ \|v\|_{L^{p^*}(\mathbb{T}^N)}\le 1}} \left|\sum_{\substack{|\alpha|\le 2l-1\\ a_\alpha\neq 0}}\int_{\mathbb{T}^N} z_\alpha(x)\, a_\alpha \mathrm{D}^\alpha(-\mathcal{A}_{p^*})^{-\delta} v(x)\,\mathrm{d}x\right|\\
&\quad= \sup_{\substack{v\in L^{p^*}(\mathbb{T}^N)\\ \|v\|_{L^{p^*}(\mathbb{T}^N)}\le 1}} \left|\int_{\mathbb{T}^N}\left\langle z(x), \left\{a_\alpha \mathrm{D}^\alpha(-\mathcal{A}_{p^*})^{-\delta} v(x)\right\}_{\substack{|\alpha|\le 2l-1\\ a_\alpha\neq 0}}\right\rangle_{\mathbb{R}^\eta}\mathrm{d}x\right|\\
&\quad\le \|z\|_{L^p(\mathbb{T}^N;\mathbb{R}^\eta)} \sup_{\substack{v\in L^{p^*}(\mathbb{T}^N)\\ \|v\|_{L^{p^*}(\mathbb{T}^N)}\le 1}} \Big\|\left\{a_\alpha \mathrm{D}^\alpha(-\mathcal{A}_{p^*})^{-\delta} v\right\}_{\substack{|\alpha|\le 2l-1\\ a_\alpha\neq 0}}\Big\|_{L^{p^*}(\mathbb{T}^N;\mathbb{R}^\eta)}\\
&\quad\le C\,\|z\|_{L^p(\mathbb{T}^N;\mathbb{R}^\eta)}
\end{aligned}
$$

and the claim follows. Next, all f_α, $|\alpha|\le 2l-1$, have bounded derivatives hence at most linear growth, so it holds for any $z\in L^p(\mathbb{T}^N)$

$$
\big\|f(z)\big\|_{L^p(\mathbb{T}^N;\mathbb{R}^\eta)}\le C\big(1+\|z\|_{L^p(\mathbb{T}^N)}\big). \tag{2.7}
$$

Later on, if there is no danger of confusion we will write $L^p(\mathbb{T}^N)$ instead of $L^p(\mathbb{T}^N;\mathbb{R}^\eta)$. Let $v\in\mathscr{H}$, then using the above remark, the fact (2.3), the estimate (2.7) and the Young inequality for convolutions we obtain

$$
\begin{aligned}
\|\mathscr{K}_1 v\|_{\mathscr{H}}^q &= \mathbb{E}\int_0^T\Big\|\int_0^t \mathcal{S}_p(t-s)F(v(s))\,\mathrm{d}s\Big\|_{L^p(\mathbb{T}^N)}^q\mathrm{d}t\\
&\le \mathbb{E}\int_0^T\left(\int_0^t\Big\|(-\mathcal{A}_p)^\delta\mathcal{S}_p(t-s)\mathcal{B}_p f(v(s))\Big\|_{L^p(\mathbb{T}^N)}\mathrm{d}s\right)^q\mathrm{d}t\\
&\le C\,\mathbb{E}\int_0^T\left(\int_0^t\frac{1}{(t-s)^\delta}\big\|\mathcal{B}_p f(v(s))\big\|_{L^p(\mathbb{T}^N)}\mathrm{d}s\right)^q\mathrm{d}t\\
&\le C\,\mathbb{E}\int_0^T\left(\int_0^t\frac{1}{(t-s)^\delta}\big\|f(v(s))\big\|_{L^p(\mathbb{T}^N)}\mathrm{d}s\right)^q\mathrm{d}t\\
&\le C\,\mathbb{E}\int_0^T\left(\int_0^t\frac{1}{(t-s)^\delta}\big(1+\|v(s)\|_{L^p(\mathbb{T}^N)}\big)\mathrm{d}s\right)^q\mathrm{d}t\\
&\le C\,T^{q(1-\delta)}\,\mathbb{E}\int_0^T\big(1+\|v(s)\|_{L^p(\mathbb{T}^N)}\big)^q\mathrm{d}s = C\,T^{q(1-\delta)}\big(T+\|v\|_{\mathscr{H}}^q\big).
\end{aligned} \tag{2.8}
$$

Next, by the Burkholder-Davis-Gundy inequality for martingales with values in 2-smooth Banach spaces (see [9], [61]), we have

$$\begin{aligned}\left\|\mathscr{K}_2 v\right\|_{\mathscr{H}}^q &= \mathbb{E}\int_0^T \left\|\int_0^t \mathcal{S}_p(t-s)\sigma(v(s))\mathrm{d}W(s)\right\|_{L^p(\mathbb{T}^N)}^q \mathrm{d}t \\ &\leq C\int_0^T \mathbb{E}\bigg(\int_0^t \left\|\mathcal{S}_p(t-s)\sigma(v(s))\right\|_{\gamma(\mathfrak{U};L^p(\mathbb{T}^N))}^2 \mathrm{d}s\bigg)^{\frac{q}{2}}\mathrm{d}t \\ &\leq C\,T^{\frac{q-2}{2}}\int_0^T \mathbb{E}\int_0^t \left\|\sigma(v(s))\right\|_{\gamma(\mathfrak{U};L^p(\mathbb{T}^N))}^q \mathrm{d}s\,\mathrm{d}t.\end{aligned} \tag{2.9}$$

The γ-radonifying norm can be computed, for almost every s and ω, using (2.5) as in (2.6). Therefore

$$\left\|\mathscr{K}_2 v\right\|_{\mathscr{H}}^q \leq C\,T^{\frac{q-2}{2}}\int_0^T \mathbb{E}\int_0^t \big(1+\|v(s)\|_{L^p(\mathbb{T}^N)}^q\big)\mathrm{d}s\,\mathrm{d}t \leq C\,T^{\frac{q}{2}}\big(T+\|v\|_{\mathscr{H}}^q\big).$$

We conclude that $\mathscr{K}(\mathscr{H})\subset\mathscr{H}$ for any $T>0$.

In order to show the contraction property of $\mathscr{K}_1$, we will mimic the procedure from (2.8) and use the Lipschitz continuity of f. Indeed, f_α, $|\alpha|\leq 2l-1$, have bounded derivatives so they are Lipschitz continuous and

$$\left\|f(z_1)-f(z_2)\right\|_{L^p(\mathbb{T}^N)} \leq C\left\|z_1-z_2\right\|_{L^p(\mathbb{T}^N)}, \qquad z_1,\,z_2\in L^p(\mathbb{T}^N),$$

can be proved as (2.7). For $v,w\in\mathscr{H}$

$$\begin{aligned}\left\|\mathscr{K}_1 v-\mathscr{K}_1 w\right\|_{\mathscr{H}}^q &= \mathbb{E}\int_0^T \left\|\int_0^t \mathcal{S}_p(t-s)\Big(F(v(s))-F(w(s))\Big)\,\mathrm{d}s\right\|_{L^p(\mathbb{T}^N)}^q \mathrm{d}t \\ &\leq \mathbb{E}\int_0^T \left(\int_0^t \left\|(-\mathcal{A}_p)^\delta \mathcal{S}_p(t-s)\mathcal{B}_p\big(f(v(s))-f(w(s))\big)\right\|_{L^p(\mathbb{T}^N)}\mathrm{d}s\right)^q \mathrm{d}t \\ &\leq C\,\mathbb{E}\int_0^T \left(\int_0^t \frac{1}{(t-s)^\delta}\left\|\mathcal{B}_p\big(f(v(s))-f(w(s)\big)\right\|_{L^p(\mathbb{T}^N)}\mathrm{d}s\right)^q \mathrm{d}t \\ &\leq C\,\mathbb{E}\int_0^T \left(\int_0^t \frac{1}{(t-s)^\delta}\left\|f(v(s))-f(w(s))\right\|_{L^p(\mathbb{T}^N)}\mathrm{d}s\right)^q \mathrm{d}t \\ &\leq C\,\mathbb{E}\int_0^T \left(\int_0^t \frac{1}{(t-s)^\delta}\left\|v(s)-w(s)\right\|_{L^p(\mathbb{T}^N)}\mathrm{d}s\right)^q \mathrm{d}t \\ &\leq C\,T^{q(1-\delta)}\,\mathbb{E}\int_0^T \left\|v(s)-w(s)\right\|_{L^p(\mathbb{T}^N)}^q \mathrm{d}s = C\,T^{q(1-\delta)}\left\|v-w\right\|_{\mathscr{H}}^q.\end{aligned}$$

In the case of $\mathscr{K}_2$ we employ the same calculations as in (2.9) and the sequel:

$$\begin{aligned}\left\|\mathscr{K}_2 v-\mathscr{K}_2 w\right\|_{\mathscr{H}}^q &= \mathbb{E}\int_0^T \left\|\int_0^t \mathcal{S}_p(t-s)\Big(\sigma(v(s))-\sigma(w(s))\Big)\mathrm{d}W(s)\right\|_{L^p(\mathbb{T}^N)}^q \mathrm{d}t \\ &\leq C\int_0^T \mathbb{E}\bigg(\int_0^t \left\|\mathcal{S}_p(t-s)\big(\sigma(v(s))-\sigma(w(s))\big)\right\|_{\gamma(\mathfrak{U};L^p(\mathbb{T}^N))}^2 \mathrm{d}s\bigg)^{\frac{q}{2}}\mathrm{d}t \\ &\leq C\,T^{\frac{q-2}{2}}\int_0^T \mathbb{E}\int_0^t \left\|\sigma(v(s))-\sigma(w(s))\right\|_{\gamma(\mathfrak{U};L^p(\mathbb{T}^N))}^q \mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

Let $z_1,\, z_2 \in L^p(\mathbb{T}^N)$. Then for the γ-radonifying norm we have

$$\begin{aligned}
&\big\|\sigma(z_1)-\sigma(z_2)\big\|^q_{\gamma(\mathfrak{U};L^p(\mathbb{T}^N))}\\
&\qquad\leq\bigg(\mathbb{E}\Big\|\sum_{i=1}^d\gamma_i\big(\sigma_i(\cdot,z_1(\cdot))-\sigma_i(\cdot,z_2(\cdot))\big)\Big\|^2_{L^p(\mathbb{T}^N)}\bigg)^{\frac{q}{2}}\\
&\qquad\leq\bigg(\mathbb{E}\Big\|\sum_{i=1}^d\gamma_i\big(\sigma_i(\cdot,z_1(\cdot))-\sigma_i(\cdot,z_2(\cdot))\big)\Big\|^p_{L^p(\mathbb{T}^N)}\bigg)^{\frac{q}{p}}\\
&\qquad=C\bigg(\int_{\mathbb{T}^N}\Big(\sum_{i=1}^d\big|\sigma_i(y,z_1(y))-\sigma_i(y,z_2(y))\big|^2\Big)^{\frac{p}{2}}\mathrm{d}y\bigg)^{\frac{q}{p}}\\
&\qquad\leq C\,\|z_1-z_2\|^q_{L^p(\mathbb{T}^N)},
\end{aligned}$$

where the last inequality follows from the fact that all σ_i, $i=1,\dots,d$, have bounded derivatives and therefore are Lipschitz continuous. We conclude

$$\big\|\mathscr{K}_2v-\mathscr{K}_2w\big\|^q_{\mathscr{H}}\leq C\,T^{\frac{q}{2}}\|v-w\|^q_{\mathscr{H}}.$$

Consequently

$$\|\mathscr{K}v-\mathscr{K}w\|_{\mathscr{H}}\leq C\,\big(T^{1-\delta}+T^{\frac{1}{2}}\big)\|v-w\|_{\mathscr{H}},$$

where the constant does not depend on T and u_0. Therefore, if

$$C\,\big(T^{1-\delta}+T^{\frac{1}{2}}\big)<1 \tag{2.10}$$

then the mapping $\mathscr{K}$ has unique fixed point u in $\mathscr{H}$ which is a mild solution of (2.1). Furthermore, by a standard use of the factorization lemma, it has continuous trajectories with values in $L^p(\mathbb{T}^N)$, i.e. belongs to

$$L^q(\Omega;C([0,T];L^p(\mathbb{T}^N))).$$

Therefore, the condition on T can be easily removed by considering the equation on intervals $[0,\tilde{T}]$, $[\tilde{T},2\tilde{T}]$, ... with $\tilde{T}$ satisfying (2.10).

□

The estimates from previous proposition can be improved in order to obtain a better regularity of u.

Proposition 2.4.2 (Estimate in $W^{1,p}(\mathbb{T}^N)$). *Let $p\in[2,\infty)$, $q\in(2,\infty)$. Assume that $u_0\in L^q(\Omega;W^{1,p}(\mathbb{T}^N))$ and*

$$f_\alpha\in C^{2l-1}(\mathbb{R}),\ \ |\alpha|\leq 2l-1;\qquad\qquad \sigma_i\in C^1(\mathbb{T}^N\times\mathbb{R}),\ \ i=1,\dots,d,$$

have bounded derivatives of first order. Then the mild solution of (2.1) belongs to

$$L^q(\Omega;C([0,T];W^{1,p}(\mathbb{T}^N)))$$

and the following estimate holds true

$$\mathbb{E}\sup_{0\leq t\leq T}\|u(t)\|^q_{W^{1,p}(\mathbb{T}^N)}\leq C\big(1+\mathbb{E}\|u_0\|^q_{W^{1,p}(\mathbb{T}^N)}\big). \tag{2.11}$$

Proof. Recall that u is the limit of Picard iterations: let $u^0(t) = u_0$ and for $n \in \mathbb{N}$ define

$$\begin{aligned} u^n(t) = \mathcal{S}_p(t)\,u_0 + \int_0^t \mathcal{S}_p(t-s)\,F\big(u^{n-1}(s)\big)\,\mathrm{d}s \\ + \int_0^t \mathcal{S}_p(t-s)\,\sigma\big(u^{n-1}(s)\big)\,\mathrm{d}W(s). \end{aligned}$$

We will show

$$\mathbb{E}\sup_{0\le t\le T}\|u^n(t)\|^q_{W^{1,p}(\mathbb{T}^N)} \le C\big(1+\mathbb{E}\|u_0\|^q_{W^{1,p}(\mathbb{T}^N)}\big), \qquad \forall n\in\mathbb{N}, \tag{2.12}$$

with a constant C independent of n. By induction on n, assume that the hypothesis is satisfied for u^{n-1} and compute the estimate for u^n. We will proceed term by term and follow the ideas of Proposition 2.4.1. Consider the operators $\mathcal{S}_p(t)$, $t\ge 0$, restricted to the Sobolev space $W^{1,p}(\mathbb{T}^N)$ and denote them by $\mathcal{S}_{1,p}(t)$, $t\ge 0$. These operators form a bounded analytic semigroup on $W^{1,p}(\mathbb{T}^N)$ generated by the part of $\mathcal{A}_p$ in $W^{1,p}(\mathbb{T}^N)$ (see [3, Theorem V.2.1.3]). Let us denote this generator by $\mathcal{A}_{1,p}$. Therefore we have

$$\begin{aligned} \mathbb{E}\sup_{0\le t\le T}\|\mathcal{S}_p(t)u_0\|^q_{W^{1,p}(\mathbb{T}^N)} &= \mathbb{E}\sup_{0\le t\le T}\|\mathcal{S}_{1,p}(t)u_0\|^q_{W^{1,p}(\mathbb{T}^N)} \\ &\le C\,\mathbb{E}\|u_0\|^q_{W^{1,p}(\mathbb{T}^N)}. \end{aligned}$$

As above, let $\delta = \frac{2l-1}{2l}$ and consider the operator

$$\begin{aligned} \mathcal{B}_{1,p}\ :\ W^{1,p}(\mathbb{T}^N;\mathbb{R}^\eta) &\longrightarrow W^{1,p}(\mathbb{T}^N) \\ \{z_\alpha\}_{\substack{|\alpha|\le 2l-1\\ a_\alpha\neq 0}} &\longmapsto (-\mathcal{A}_p)^{-\delta}\sum_{\substack{|\alpha|\le 2l-1\\ a_\alpha\neq 0}} a_\alpha \mathrm{D}^\alpha z_\alpha. \end{aligned}$$

We will show that it is a bounded operator. Indeed, according to the computations in the proof of Proposition 2.4.1, for any $z\in W^{1,p}(\mathbb{T}^N;\mathbb{R}^\eta)$,

$$\big\|\mathcal{B}_{1,p}z\big\|_{L^p(\mathbb{T}^N)} \le C\|z\|_{L^p(\mathbb{T}^N;\mathbb{R}^\eta)}.$$

For any multiindex $\beta = (\beta_1,\dots,\beta_N)$ such that $|\beta| = 1$, we can write

$$\big\|\mathrm{D}^\beta\mathcal{B}_{1,p}z\big\|_{L^p(\mathbb{T}^N)} = \Big\|\mathrm{D}^\beta(-\mathcal{A}_p)^{-\frac{1}{2l}}(-\mathcal{A}_p)^{-\frac{2l-1}{2l}+\frac{1}{2l}}\sum_{\substack{|\alpha|\le 2l-1\\ a_\alpha\neq 0}} a_\alpha\mathrm{D}^\alpha z_\alpha\Big\|_{L^p(\mathbb{T}^N)},$$

where the operator $L^p(\mathbb{T}^N)\to L^p(\mathbb{T}^N)$, $v\mapsto \mathrm{D}^\beta(-\mathcal{A}_p)^{-\frac{1}{2l}}v$, is bounded. For each α, $|\alpha|\le 2l-1$, let us fix a multiindex α' such that it is of order 1 and $\alpha-\alpha'$ is also a multiindex, i.e. $|\alpha'| = 1$ and $|\alpha-\alpha'| = |\alpha|-1$. Note, that if p^* is the conjugate exponent to p, the operator $L^{p^*}(\mathbb{T}^N)\to L^{p^*}(\mathbb{T}^N)$, $v\mapsto a_\alpha\mathrm{D}^{\alpha-\alpha'}(-\mathcal{A}_{p^*})^{\frac{-2l+2}{2l}}v$, $|\alpha|\le 2l-1$, is bounded as well. We conclude

$$
\begin{aligned}
&\Big\|(-\mathcal{A}_p)^{\frac{-2l+2}{2l}}\sum_{\substack{|\alpha|\leq 2l-1\\ a_\alpha\neq 0}} a_\alpha \mathrm{D}^\alpha z_\alpha\Big\|_{L^p(\mathbb{T}^N)}\\
&\quad= \sup_{\substack{v\in L^{p^*}(\mathbb{T}^N)\\ \|v\|_{L^{p^*}(\mathbb{T}^N)}\leq 1}}\left|\int_{\mathbb{T}^N}(-\mathcal{A}_p)^{\frac{-2l+2}{2l}}\sum_{\substack{|\alpha|\leq 2l-1\\ a_\alpha\neq 0}} a_\alpha \mathrm{D}^\alpha z_\alpha(x)\,v(x)\,\mathrm{d}x\right|\\
&\quad= \sup_{\substack{v\in L^{p^*}(\mathbb{T}^N)\\ \|v\|_{L^{p^*}(\mathbb{T}^N)}\leq 1}}\left|\sum_{\substack{|\alpha|\leq 2l-1\\ a_\alpha\neq 0}}\int_{\mathbb{T}^N}\mathrm{D}^{\alpha'} z_\alpha(x)\,a_\alpha \mathrm{D}^{\alpha-\alpha'}(-\mathcal{A}_{p^*})^{\frac{-2l+2}{2l}}v(x)\,\mathrm{d}x\right|\\
&\quad\leq \Big\|\big\{\mathrm{D}^{\alpha'}z_\alpha\big\}_{\substack{|\alpha|\leq 2l-1\\ a_\alpha\neq 0}}\Big\|_{L^p(\mathbb{T}^N;\mathbb{R}^\eta)}\\
&\qquad\times\sup_{\substack{v\in L^{p^*}(\mathbb{T}^N)\\ \|v\|_{L^{p^*}(\mathbb{T}^N)}\leq 1}}\Big\|\Big\{a_\alpha \mathrm{D}^{\alpha-\alpha'}(-\mathcal{A}_{p^*})^{\frac{-2l+2}{2l}}v\Big\}_{\substack{|\alpha|\leq 2l-1\\ a_\alpha\neq 0}}\Big\|_{L^{p^*}(\mathbb{T}^N;\mathbb{R}^\eta)}\\
&\quad\leq C\,\|z\|_{W^{1,p}(\mathbb{T}^N;\mathbb{R}^\eta)}
\end{aligned}
$$

and the claim follows. Therefore, we have

$$
\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq T}\Big\|\int_0^t\mathcal{S}_p(t-s)F\big(u^{n-1}(s)\big)\,\mathrm{d}s\Big\|^q_{W^{1,p}(\mathbb{T}^N)}\\
&\quad\leq \mathbb{E}\sup_{0\leq t\leq T}\left(\int_0^t\Big\|(-\mathcal{A}_p)^\delta\mathcal{S}_p(t-s)\mathcal{B}_{1,p}f\big(u^{n-1}(s)\big)\Big\|_{W^{1,p}(\mathbb{T}^N)}\mathrm{d}s\right)^q\\
&\quad\leq \mathbb{E}\sup_{0\leq t\leq T}\left(\int_0^t\Big\|(-\mathcal{A}_{1,p})^\delta\mathcal{S}_{1,p}(t-s)\mathcal{B}_{1,p}f\big(u^{n-1}(s)\big)\Big\|_{W^{1,p}(\mathbb{T}^N)}\mathrm{d}s\right)^q\\
&\quad\leq C\,\mathbb{E}\sup_{0\leq t\leq T}\left(\int_0^t\frac{1}{(t-s)^\delta}\big\|f\big(u^{n-1}(s)\big)\big\|_{W^{1,p}(\mathbb{T}^N)}\mathrm{d}s\right)^q\\
&\quad\leq CT^{q(1-\delta)}\mathbb{E}\sup_{0\leq t\leq T}\big\|f\big(u^{n-1}(t)\big)\big\|^q_{W^{1,p}(\mathbb{T}^N)}.
\end{aligned}
$$

To deduce a similar estimate for the stochastic term, we need to consider stochastic integration in $W^{1,p}(\mathbb{T}^N)$. Employing the Hölder inequality and the equivalence of norms on $W^{1,p}(\mathbb{T}^N)$ we obtain for $z \in W^{1,p}(\mathbb{T}^N)$

$$\begin{aligned}
\big\|\sigma(z)\big\|^q_{\gamma(\mathfrak{U};W^{1,p}(\mathbb{T}^N))} &= \bigg(\mathbb{E}\Big\|\sum_{i=1}^d \gamma_i\,\sigma_i\big(\cdot,z(\cdot)\big)\Big\|^2_{W^{1,p}(\mathbb{T}^N)}\bigg)^{\frac{q}{2}} \\
&\leq \bigg(\mathbb{E}\Big\|\sum_{i=1}^d \gamma_i\,\sigma_i\big(\cdot,z(\cdot)\big)\Big\|^p_{W^{1,p}(\mathbb{T}^N)}\bigg)^{\frac{q}{p}} \\
&\leq C\bigg(\mathbb{E}\Big\|\sum_{i=1}^d \gamma_i\,(-\mathcal{A}_p)^{\frac{1}{2l}}\sigma_i\big(\cdot,z(\cdot)\big)\Big\|^p_{L^p(\mathbb{T}^N)}\bigg)^{\frac{q}{p}} \\
&= C\bigg(\int_{\mathbb{T}^N}\Big(\sum_{i=1}^d \big|(-\mathcal{A}_p)^{\frac{1}{2l}}\sigma_i\big(y,z(y)\big)\big|^2\Big)^{\frac{p}{2}}\mathrm{d}y\bigg)^{\frac{q}{p}} \\
&\leq C\sum_{i=1}^d \big\|\sigma_i\big(\cdot,z(\cdot)\big)\big\|^q_{W^{1,p}(\mathbb{T}^N)}.
\end{aligned}$$

Since $q \in (2,\infty)$, we make use of the maximal estimate for stochastic convolution [9, Corollary 3.5] which can be proved by the factorization method. For the reader's convenience we recall the basic steps of the proof. Let $\vartheta \in (1/q, 1/2)$, then according to the stochastic Fubini theorem [10, Proposition 3.3(v)],

$$\int_0^t \mathcal{S}_p(t-s)\sigma\big(u^{n-1}(s)\big)\mathrm{d}W(s) = \frac{1}{\Gamma(\vartheta)}\int_0^t (t-s)^{\vartheta-1}\mathcal{S}_p(t-s)\,y(s)\mathrm{d}s,$$

where

$$y(s) = \frac{1}{\Gamma(1-\vartheta)}\int_0^s (s-r)^{-\vartheta}\mathcal{S}_p(s-r)\sigma\big(u^{n-1}(r)\big)\mathrm{d}W(r).$$

Hence application of the Hölder, Burkholder-Davis-Gundy and Young inequalities yields (here the constant C is independent on T)

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq T}\bigg\|\int_0^t \mathcal{S}_p(t-s)\sigma\big(u^{n-1}(s)\big)\mathrm{d}W(s)\bigg\|^q_{W^{1,p}(\mathbb{T}^N)} \\
&\qquad\leq CT^{\frac{q}{2}-1}\,\mathbb{E}\int_0^T \big\|\sigma\big(u^{n-1}(s)\big)\big\|^q_{\gamma(\mathfrak{U};W^{1,p}(\mathbb{T}^N))}\mathrm{d}s
\end{aligned}$$

so

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq T}\bigg\|\int_0^t \mathcal{S}_p(t-s)\sigma\big(u^{n-1}(s)\big)\mathrm{d}W(s)\bigg\|^q_{W^{1,p}(\mathbb{T}^N)} \\
&\qquad\leq CT^{\frac{q}{2}-1}\sum_{i=1}^d \mathbb{E}\int_0^T \big\|\sigma_i\big(\cdot,u^{n-1}(s,\cdot)\big)\big\|^q_{W^{1,p}(\mathbb{T}^N)}\mathrm{d}s \\
&\qquad\leq CT^{\frac{q}{2}}\sum_{i=1}^d \mathbb{E}\sup_{0\leq t\leq T}\big\|\sigma_i\big(\cdot,u^{n-1}(t,\cdot)\big)\big\|^q_{W^{1,p}(\mathbb{T}^N)}
\end{aligned}$$

and finally

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\|u^n(t)\|^q_{W^{1,p}(\mathbb{T}^N)}\le C\,\mathbb{E}\|u_0\|^q_{W^{1,p}(\mathbb{T}^N)}&\\+CT^{q(1-\delta)}\mathbb{E}\sup_{0\le t\le T}\big\|f\big(u^{n-1}(t)\big)\big\|^q_{W^{1,p}(\mathbb{T}^N)}&\\+CT^{\frac{q}{2}}\sum_{i=1}^d\mathbb{E}\sup_{0\le t\le T}\big\|\sigma_i\big(\cdot,u^{n-1}(t,\cdot)\big)\big\|^q_{W^{1,p}(\mathbb{T}^N)},&\end{aligned}$$

where the constant does not depend on n. Now, we make use of Remark 2.3.3 and obtain

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\|u^n(t)\|^q_{W^{1,p}(\mathbb{T}^N)}\le C\,\mathbb{E}\|u_0\|^q_{W^{1,p}(\mathbb{T}^N)}&\\+C\big(T^{q(1-\delta)}+T^{\frac{q}{2}}\big)\bigg(1+\mathbb{E}\sup_{0\le t\le T}\|u^{n-1}(t)\|^q_{W^{1,p}(\mathbb{T}^N)}\bigg).&\end{aligned}$$

Let us make an additional hypothesis: assume that T is such that

$$C_T=C\big(T^{q(1-\delta)}+T^{\frac{q}{2}}\big)<1. \tag{2.13}$$

Denoting $K_n=\mathbb{E}\sup_{0\le t\le T}\|u^n(t)\|^q_{W^{1,p}(\mathbb{T}^N)}$, $n\in\mathbb{N}_0$, we have

$$K_n\le C\,\mathbb{E}\|u_0\|^q_{W^{1,p}(\mathbb{T}^N)}+C_T\big(1+K_{n-1}\big)$$

and inductively in n

$$\mathbb{E}\sup_{0\le t\le T}\|u^n(t)\|^q_{W^{1,p}(\mathbb{T}^N)}\le\tilde{C}_T\big(1+\mathbb{E}\|u_0\|^q_{W^{1,p}(\mathbb{T}^N)}\big), \tag{2.14}$$

where $\tilde{C}_T$ is independent n. So (2.12) follows if T is sufficiently small.

In order to remove this condition, we consider a suitable partition of the interval $[0,T]$. Let $\tilde{T}>0$ satisfy (2.13) and $0<\tilde{T}<2\tilde{T}<\cdots<K\tilde{T}=T$ for some $K\in\mathbb{N}$. Fix $k\in\{1,\dots,K\}$. We will study the processes u^n, $n\in\mathbb{N}$, on the interval $[(k-1)\tilde{T},k\tilde{T}]$ and find an estimate similar to (2.14). Each u^n, $n\in\mathbb{N}$, is the unique mild solution to the corresponding linear equation

$$\begin{aligned}&\mathrm{d}u^n=\big[\mathcal{A}u^n+F\big(u^{n-1}\big)\big]\,\mathrm{d}t+\sigma\big(u^{n-1}\big)\,\mathrm{d}W, \qquad x\in\mathbb{T}^N,\ t\in(0,T),\\&u(0)=u_0.\end{aligned}$$

Let $v(t,s,;u_0)$, $t\ge s\ge 0$, be the mild solution of this problem with the initial condition u_0 given at time s. It follows from the uniqueness that for arbitrary $t\ge r\ge s\ge 0$

$$v\big(t,r;v(r,s;u_0)\big)=v(t,s;u_0)\qquad \mathbb{P}\text{-a.s.}$$

and therefore we can write

$$\begin{aligned}u^n(t)=&\,\mathcal{S}_p\big(t-(k-1)\tilde{T}\big)u^n\big((k-1)\tilde{T}\big)+\int_{(k-1)\tilde{T}}^t\mathcal{S}_p(t-s)F\big(u^{n-1}(s)\big)\,\mathrm{d}s\\&+\int_{(k-1)\tilde{T}}^t\mathcal{S}_p(t-s)\sigma\big(u^{n-1}(s)\big)\,\mathrm{d}W(s),\qquad t\in\big[(k-1)\tilde{T},T\big].\end{aligned}$$

Following the same approach as above we obtain

$$\mathbb{E}\sup_{(k-1)\tilde{T}\le t\le k\tilde{T}}\|u^n(t)\|^q_{W^{1,p}(\mathbb{T}^N)}\le \tilde{C}_{\tilde{T}}\Big(1+\mathbb{E}\big\|u^n\big((k-1)\tilde{T}\big)\big\|^q_{W^{1,p}(\mathbb{T}^N)}\Big)$$

with a constant similar to $\tilde{C}_T$ in (2.14). Hence

$$\begin{aligned}\mathbb{E}&\sup_{(k-1)\tilde{T}\le t\le k\tilde{T}}\|u^n(t)\|^q_{W^{1,p}(\mathbb{T}^N)}\\&\le \tilde{C}_{\tilde{T}}\Big(1+\mathbb{E}\sup_{(k-2)\tilde{T}\le t\le (k-1)\tilde{T}}\|u^n(t)\|^q_{W^{1,p}(\mathbb{T}^N)}\Big)\\&\le\sum_{i=1}^K(\tilde{C}_{\tilde{T}})^i+(\tilde{C}_{\tilde{T}})^K\mathbb{E}\|u_0\|^q_{W^{1,p}(\mathbb{T}^N)}\le\bar{C}\big(1+\mathbb{E}\|u_0\|^q_{W^{1,p}(\mathbb{T}^N)}\big),\end{aligned}$$

where the constant $\bar{C}$ is independent of k and n. Finally, the estimate (2.12) follows:

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\|u^n(t)\|^q_{W^{1,p}(\mathbb{T}^N)}&=\mathbb{E}\max_{k=1,\dots,K}\sup_{(k-1)\tilde{T}\le t\le k\tilde{T}}\|u^n(t)\|^q_{W^{1,p}(\mathbb{T}^N)}\\&\le\sum_{k=1}^K\mathbb{E}\sup_{(k-1)\tilde{T}\le t\le k\tilde{T}}\|u^n(t)\|^q_{W^{1,p}(\mathbb{T}^N)}\le K\bar{C}\big(1+\mathbb{E}\|u_0\|^q_{W^{1,p}(\mathbb{T}^N)}\big).\end{aligned}$$

We have now all in hand to deduce that the sequence $\{u^n;\,n\in\mathbb{N}\}$ is bounded in

$$L^q(\Omega;L^\infty(0,T;W^{1,p}(\mathbb{T}^N)))$$

and therefore has a weak-star convergent subsequence. Since any norm is weak-star lower semicontinuous we get the estimate (2.11) for the limit process u. Moreover, since the stochastic convolution has a continuous modification according to [9, Corollary 3.5], the proof is complete. □

Proof of regularity in higher order Sobolev spaces (order greater than 1) is more complicated as the norm of a superposition does not, in general, grow linearly with the norm of the inner function (cf. Proposition 2.3.1, Corollary 2.3.2, Remark 2.3.3).

Proposition 2.4.3 (Estimate in $W^{m,p}(\mathbb{T}^N)$)**.** *Let $p\in[2,\infty)$, $q\in(2,\infty)$, $m\in\mathbb{N}$, $m\ge2$. Assume that $u_0\in L^q(\Omega;W^{m,p}(\mathbb{T}^N))\cap L^{mq}(\Omega;W^{1,mp}(\mathbb{T}^N))$ and*

$$f_\alpha\in C^m(\mathbb{R})\cap C^{2l-1}(\mathbb{R}),\ \ |\alpha|\le 2l-1;\qquad \sigma_i\in C^m(\mathbb{T}^N\times\mathbb{R}),\ \ i=1,\dots,d,$$

have bounded derivatives up to order m. Then the mild solution of (2.1) belongs to

$$L^q(\Omega;C([0,T];W^{m,p}(\mathbb{T}^N)))$$

and the following estimate holds true

$$\mathbb{E}\sup_{0\le t\le T}\|u(t)\|^q_{W^{m,p}(\mathbb{T}^N)}\le C\big(1+\mathbb{E}\|u_0\|^q_{W^{m,p}(\mathbb{T}^N)}+\mathbb{E}\|u_0\|^{mq}_{W^{1,mp}(\mathbb{T}^N)}\big).\tag{2.15}$$

Proof. First, we intend to prove the following estimate for the Picard iterations

$$\mathbb{E}\sup_{0\le t\le T}\|u^n(t)\|^q_{W^{m,p}(\mathbb{T}^N)}\le C\big(1+\mathbb{E}\|u_0\|^q_{W^{m,p}(\mathbb{T}^N)}+\mathbb{E}\|u_0\|^{mq}_{W^{1,mp}(\mathbb{T}^N)}\big),\tag{2.16}$$

with a constant independent of n. By induction on n, assume that the hypothesis is satisfied for u^{n-1} and compute the estimate for u^n. The following arguments and calculations are mostly similar to those in Proposition 2.4.2. Recall that according to (2.12), we have

$$\mathbb{E}\sup_{0\le t\le T}\|u^n(t)\|^{mq}_{W^{1,mp}(\mathbb{T}^N)}\le C\big(1+\mathbb{E}\|u_0\|^{mq}_{W^{1,mp}(\mathbb{T}^N)}\big),\qquad \forall n\in\mathbb{N}. \tag{2.17}$$

Let us consider the restrictions of the operators $\mathcal{S}_p(t)$, $t\ge 0$, to the Sobolev space $W^{m,p}(\mathbb{T}^N)$ and denote them by $\mathcal{S}_{m,p}(t)$, $t\ge 0$. By [3, Theorem V.2.1.3], we obtain a strongly continuous semigroup on $W^{m,p}(\mathbb{T}^N)$ generated by part of $\mathcal{A}_p$ in $W^{m,p}(\mathbb{T}^N)$. We denote the generator by $\mathcal{A}_{m,p}$. It follows

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\|\mathcal{S}_p(t)u_0\|^q_{W^{m,p}(\mathbb{T}^N)}&=\mathbb{E}\sup_{0\le t\le T}\|\mathcal{S}_{m,p}(t)u_0\|^q_{W^{m,p}(\mathbb{T}^N)}\\&\le C\,\mathbb{E}\|u_0\|^q_{W^{m,p}(\mathbb{T}^N)}.\end{aligned}$$

As above, we employ the following bounded operator: let $\delta=\frac{2l-1}{2l}$

$$\begin{aligned}\mathcal{B}_{m,p}\ :\ W^{m,p}(\mathbb{T}^N;\mathbb{R}^\eta)&\longrightarrow W^{m,p}(\mathbb{T}^N)\\ \{z_\alpha\}_{\substack{|\alpha|\le 2l-1\\ a_\alpha\neq 0}}&\longmapsto(-\mathcal{A}_p)^{-\delta}\sum_{\substack{|\alpha|\le 2l-1\\ a_\alpha\neq 0}}a_\alpha\mathrm{D}^\alpha z_\alpha,\end{aligned}$$

so

$$\begin{aligned}&\mathbb{E}\sup_{0\le t\le T}\bigg\|\int_0^t\mathcal{S}_p(t-s)F\big(u^{n-1}(s)\big)\,\mathrm{d}s\bigg\|^q_{W^{m,p}(\mathbb{T}^N)}\\&\quad\le\mathbb{E}\sup_{0\le t\le T}\bigg(\int_0^t\Big\|(-\mathcal{A}_{m,p})^\delta\mathcal{S}_{m,p}(t-s)\mathcal{B}_{m,p}f\big(u^{n-1}(s)\big)\Big\|_{W^{m,p}(\mathbb{T}^N)}\mathrm{d}s\bigg)^q\\&\quad\le C\,\mathbb{E}\sup_{0\le t\le T}\bigg(\int_0^t\frac{1}{(t-s)^\delta}\big\|f\big(u^{n-1}(s)\big)\big\|_{W^{m,p}(\mathbb{T}^N)}\mathrm{d}s\bigg)^q\\&\quad\le CT^{q(1-\delta)}\mathbb{E}\sup_{0\le t\le T}\big\|f\big(u^{n-1}(t)\big)\big\|^q_{W^{m,p}(\mathbb{T}^N)}.\end{aligned}$$

And for the stochastic term, $z\in W^{m,p}(\mathbb{T}^N)$,

$$\begin{aligned}\|\sigma(z)\|^q_{\gamma(\mathfrak{U};W^{m,p}(\mathbb{T}^N))}&=\bigg(\mathbb{E}\Big\|\sum_{i=1}^d\gamma_i\,\sigma_i\big(\cdot,z(\cdot)\big)\Big\|^2_{W^{m,p}(\mathbb{T}^N)}\bigg)^{\frac{q}{2}}\\&\le C\bigg(\mathbb{E}\Big\|\sum_{i=1}^d\gamma_i\,(-\mathcal{A}_p)^{\frac{m}{2l}}\sigma_i\big(\cdot,z(\cdot)\big)\Big\|^p_{L^p(\mathbb{T}^N)}\bigg)^{\frac{q}{p}}\\&=C\bigg(\int_{\mathbb{T}^N}\Big(\sum_{i=1}^d\big|(-\mathcal{A}_p)^{\frac{m}{2l}}\sigma_i\big(y,z(y)\big)\big|^2\Big)^{\frac{p}{2}}\mathrm{d}y\bigg)^{\frac{q}{p}}\\&\le C\sum_{i=1}^d\big\|\sigma_i\big(\cdot,z(\cdot)\big)\big\|^q_{W^{m,p}(\mathbb{T}^N)}\end{aligned}$$

hence

$$
\begin{aligned}
\mathbb{E}\sup_{0\le t\le T}&\left\|\int_0^t \mathcal{S}_p(t-s)\sigma\big(u^{n-1}(s)\big)\mathrm{d}W(s)\right\|^q_{W^{m,p}(\mathbb{T}^N)}\\
&\le CT^{\frac{q}{2}-1}\,\mathbb{E}\int_0^T \big\|\sigma\big(u^{n-1}(s)\big)\big\|^q_{\gamma(\mathfrak{U};W^{m,p}(\mathbb{T}^N))}\mathrm{d}s\\
&\le CT^{\frac{q}{2}}\sum_{i=1}^d \mathbb{E}\sup_{0\le t\le T}\big\|\sigma_i\big(\cdot,u^{n-1}(t,\cdot)\big)\big\|^q_{W^{m,p}(\mathbb{T}^N)}.
\end{aligned}
$$

We conclude

$$
\begin{aligned}
\mathbb{E}\sup_{0\le t\le T}\|u^n(t)\|^q_{W^{m,p}(\mathbb{T}^N)} &\le C\,\mathbb{E}\|u_0\|^q_{W^{m,p}(\mathbb{T}^N)}\\
&+CT^{q(1-\delta)}\mathbb{E}\sup_{0\le t\le T}\big\|f\big(u^{n-1}(t)\big)\big\|^q_{W^{m,p}(\mathbb{T}^N)}\\
&+CT^{\frac{q}{2}}\sum_{i=1}^d\mathbb{E}\sup_{0\le t\le T}\big\|\sigma_i\big(\cdot,u^{n-1}(t,\cdot)\big)\big\|^q_{W^{m,p}(\mathbb{T}^N)}.
\end{aligned}
$$

Applying Proposition 2.3.1, Corollary 2.3.2 and (2.17) we obtain

$$
\begin{aligned}
\mathbb{E}\sup_{0\le t\le T}&\|u^n(t)\|^q_{W^{m,p}(\mathbb{T}^N)}\le C\,\mathbb{E}\|u_0\|^q_{W^{m,p}(\mathbb{T}^N)}+C\big(T^{q(1-\delta)}+T^{\frac{q}{2}}\big)\\
&\times\Big(1+\mathbb{E}\sup_{0\le t\le T}\|u^{n-1}(t)\|^{mq}_{W^{1,mp}(\mathbb{T}^N)}+\mathbb{E}\sup_{0\le t\le T}\|u^{n-1}(t)\|^q_{W^{m,p}(\mathbb{T}^N)}\Big)\\
\le{}& C\,\mathbb{E}\|u_0\|^q_{W^{m,p}(\mathbb{T}^N)}+C\big(T^{q(1-\delta)}+T^{\frac{q}{2}}\big)\\
&\times\Big(1+\mathbb{E}\|u_0\|^{mq}_{W^{1,mp}(\mathbb{T}^N)}+\mathbb{E}\sup_{0\le t\le T}\|u^{n-1}(t)\|^q_{W^{m,p}(\mathbb{T}^N)}\Big).
\end{aligned}
$$

Let T satisfy the following condition

$$
C_T=C\big(T^{q(1-\delta)}+T^{\frac{q}{2}}\big)<1
$$

and define $K_n=\mathbb{E}\sup_{0\le t\le T}\|u^n(t)\|^q_{W^{m,p}(\mathbb{T}^N)}$, $n\in\mathbb{N}_0$, $L_0=\mathbb{E}\|u_0\|^{mq}_{W^{1,mp}(\mathbb{T}^N)}$. Then we have

$$
K_n\le C\,\mathbb{E}\|u_0\|^q_{W^{m,p}(\mathbb{T}^N)}+C_T\big(1+L_0+K_{n-1}\big)
$$

hence inductively in n

$$
\mathbb{E}\sup_{0\le t\le T}\|u^n(t)\|^q_{W^{m,p}(\mathbb{T}^N)}\le\tilde{C}_T\big(1+\mathbb{E}\|u_0\|^q_{W^{m,p}(\mathbb{T}^N)}+\mathbb{E}\|u_0\|^{mq}_{W^{1,mp}(\mathbb{T}^N)}\big),
$$

where the constant does not depend on n. Therefore (2.16) follows under the additional hypothesis upon T. However, this condition can be removed by the same approach as in Proposition 2.4.2.

Similarly to Proposition 2.4.2 we deduce that the sequence $\{u^n;\,n\in\mathbb{N}\}$ is bounded in

$$
L^q(\Omega;L^\infty(0,T;W^{m,p}(\mathbb{T}^N)))
$$

and therefore (2.15) holds true. Existence of a continuous modification follows again from [9, Corollary 3.5]. □

Proof of Theorem 2.2.1. If $m = 1$ the proof is an immediate consequence of Propositions 2.4.1 and 2.4.2. The case $m \geq 2$ follows from Propositions 2.4.1, 2.4.2 and 2.4.3. $\square$

Proof of Corollary 2.2.2. Let $m = k + 1$. According to Theorem 2.2.1 there exists a solution of (2.1) which belongs to

$$L^q(\Omega; C([0,T]; W^{m,p}(\mathbb{T}^N))), \qquad \forall p \in [2, \infty).$$

If $p > N$, then according to the Sobolev embedding theorem, the space $W^{m,p}(\mathbb{T}^N)$ is continuously embedded in $C^{k,\lambda}(\mathbb{T}^N)$ for $\lambda \in (0, 1 - N/p)$. Hence the assertion follows. $\square$

Chapter 3

Degenerate Parabolic Stochastic Partial Differential Equations

Abstract: We study the Cauchy problem for a scalar semilinear degenerate parabolic partial differential equation with stochastic forcing. In particular, we are concerned with the well-posedness in any space dimension. We adapt the notion of kinetic solution which is well suited for degenerate parabolic problems and supplies a good technical framework to prove the comparison principle. The proof of existence is based on the vanishing viscosity method: the solution is obtained by a compactness argument as the limit of solutions of nondegenerate approximations.

Results of this chapter are available as a preprint:

- M. Hofmanová, *Degenerate Parabolic Stochastic Partial Differential Equations.*

3.1 Introduction

In this paper, we study the Cauchy problem for a scalar semilinear degenerate parabolic partial differential equation with stochastic forcing

$$\begin{aligned} \mathrm{d}u + \mathrm{div}\,\big(B(u)\big)\mathrm{d}t &= \mathrm{div}\,\big(A(x)\nabla u\big)\mathrm{d}t + \varPhi(u)\,\mathrm{d}W, \quad x\in\mathbb{T}^N,\ t\in(0,T), \\ u(0) &= u_0, \end{aligned} \tag{3.1}$$

where W is a cylindrical Wiener process. Equations of this type are widely used in fluid mechanics since they model the phenomenon of convection-diffusion of ideal fluid in porous media. Namely, the important applications including for instance two or three-phase flows can be found in petroleum engineering or in hydrogeology. For a thorough exposition of this area given from a practical point of view we refer the reader to [26] and to the references cited therein.

The aim of the present paper is to establish the well-posedness theory for solutions of the Cauchy problem (3.1) in any space dimension. Towards this end, we adapt the notion of kinetic formulation and kinetic solution which has already been studied in the case of hyperbolic scalar conservation laws in both deterministic (see e.g. [38], [55], [56], [63], or [64] for a general presentation) and stochastic setting (see [16]); and also in the case of deterministic degenerate parabolic equations of second-order (see [13]). To the best of our knowledge, in the degenerate case, stochastic equations of type (3.1) have not been studied yet, neither by means of kinetic formulation nor by any other approach.

The concept of kinetic solution was first introduced by Lions, Perthame, Tadmor in [56] for deterministic scalar conservation laws and applies to more general situations than the one of entropy solution as considered for example in [12], [20], [45]. Moreover, it appears to be better suited particularly for degenerate parabolic problems since it allows us to keep the precise structure of the parabolic dissipative measure, whereas in the case of entropy solution part of this information is lost and has to be recovered at some stage. This technique also supplies a good technical framework to prove the L^1-comparison principle which allows to prove uniqueness. Nevertheless, kinetic formulation can be derived only for smooth solutions hence the classical result [30] giving L^p-valued solutions for the nondegenerate case has to be improved (see [32], [20]).

In the case of hyperbolic scalar conservation laws, Debussche and Vovelle [16] defined a notion of generalized kinetic solution and obtained a comparison result showing that any generalized kinetic solution is actually a kinetic solution. Accordingly, the proof of existence simplified since only weak convergence of approximate viscous solutions was necessary. The situation is quite different in the case of parabolic scalar conservation laws. Indeed, due to the parabolic term, we are not able to apply the approach of [16]: we prove the comparison principle for kinetic solutions only (not generalized ones) and therefore strong convergence of approximate solutions is needed in order to prove the existence. Moreover, the proof of the comparison principle itself is much more delicate than in the hyperbolic case.

We note that an important step in the proof of existence, identification of the limit of an approximating sequence of solutions, is based on a new general method of constructing martingale solutions of SPDEs (see Propositions 3.4.14, 3.4.15 and the sequel), that does not rely on any kind of martingale representation theorem and therefore holds independent interest especially in situations where these representation theorems are no longer available. First applications were already done in [11], [60] and, in the finite-dimensional case, also in [34]. In the present work, this method is further generalized as the martingales to be dealt with are only defined for almost all times.

The exposition is organised as follows. In Section 3.2 we review the basic setting and define the notion of kinetic solution. Section 3.3 is devoted to the proof of uniqueness. We first establish a technical Proposition 3.3.2 which then turns out to be the keystone in the proof of comparison principle in Theorem 3.3.3. We next turn to the proof of existence in Sections 3.4 and 3.5. First of all, in Section 3.4, we make an additional hypothesis upon the initial condition and employ the vanishing viscosity method. In particular, we study certain nondegenerate problems and establish suitable uniform estimates for the corresponding sequence of approximate solutions. The compactness argument then yields the existence of a martingale kinetic solution which together with the pathwise uniqueness gives the desired kinetic solution (defined on the original stochastic basis). In Section 3.5, the existence of a kinetic solution is shown for general initial data. In the final section 3.A, we formulate and prove an auxiliary result concerning densely defined martingales.

3.2 Notation and main result

We now give the precise assumptions on each of the terms appearing in the above equation (3.1). We work on a finite-time interval $[0,T]$, $T>0$, and consider periodic boundary conditions: $x\in\mathbb{T}^N$ where $\mathbb{T}^N$ is the N-dimensional torus. The flux function

$$B=(B_1,\dots,B_N):\mathbb{R}\longrightarrow\mathbb{R}^N$$

is supposed to be of class C^1 with a polynomial growth of its derivative, which is denoted by $b=(b_1,\dots,b_N)$. The diffusion matrix

$$A=(A_{ij})_{i,j=1}^N:\mathbb{T}^N\longrightarrow\mathbb{R}^{N\times N}$$

is of class C^∞, symmetric and positive semidefinite. Its square-root matrix, which is also symmetric and positive semidefinite, is denoted by σ.

Regarding the stochastic term, let $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq0},\mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration. Let $\mathcal{P}$ denote the predictable σ-algebra on $\Omega\times[0,T]$ associated to $(\mathscr{F}_t)_{t\geq0}$. The initial datum may be random in general, i.e. $\mathscr{F}_0$-measurable, and we assume $u_0\in L^p(\Omega;L^p(\mathbb{T}^N))$ for all $p\in[1,\infty)$. The process W is a cylindrical Wiener process: $W(t)=\sum_{k\geq1}\beta_k(t)e_k$ with $(\beta_k)_{k\geq1}$ being mutually independent real-valued standard Wiener processes relative to $(\mathscr{F}_t)_{t\geq0}$ and $(e_k)_{k\geq1}$ a complete orthonormal system in a separable Hilbert space $\mathfrak{U}$. In this setting, we can assume, without loss of generality, that the σ-algebra $\mathscr{F}$ is countably generated and $(\mathscr{F}_t)_{t\geq0}$ is the filtration generated by the Wiener process and the initial condition. For each $z\in L^2(\mathbb{T}^N)$ we consider a mapping $\varPhi(z):\mathfrak{U}\to L^2(\mathbb{T}^N)$ defined by $\varPhi(z)e_k=g_k(\cdot,z(\cdot))$. In particular, we suppose that $g_k\in C(\mathbb{T}^N\times\mathbb{R})$ and the following conditions

$$G^2(x,\xi)=\sum_{k\geq1}\big|g_k(x,\xi)\big|^2\leq L\big(1+|\xi|^2\big),\tag{3.2}$$

$$\sum_{k\geq1}\big|g_k(x,\xi)-g_k(y,\zeta)\big|^2\leq L\big(|x-y|^2+|\xi-\zeta|h(|\xi-\zeta|)\big),\tag{3.3}$$

are fulfilled for every $x,y\in\mathbb{T}^N$, $\xi,\zeta\in\mathbb{R}$, where h is a continuous nondecreasing function on $\mathbb{R}_+$ satisfying, for some $\alpha>0$,

$$h(\delta)\leq C\delta^\alpha,\quad \delta<1.\tag{3.4}$$

The conditions imposed on Φ, particularly assumption (3.2), imply that

$$\Phi : L^2(\mathbb{T}^N) \longrightarrow L_2(\mathfrak{U}; L^2(\mathbb{T}^N)),$$

where $L_2(\mathfrak{U}; L^2(\mathbb{T}^N))$ denotes the collection of Hilbert-Schmidt operators from $\mathfrak{U}$ to $L^2(\mathbb{T}^N)$. Thus, given a predictable process $u \in L^2(\Omega; L^2(0,T; L^2(\mathbb{T}^N)))$, the stochastic integral $t \mapsto \int_0^t \Phi(u)\mathrm{d}W$ is a well defined process taking values in $L^2(\mathbb{T}^N)$ (see [15] for detailed construction).

Finally, define the auxiliary space $\mathfrak{U}_0 \supset \mathfrak{U}$ via

$$\mathfrak{U}_0 = \bigg\{ v = \sum_{k\geq 1} \alpha_k e_k;\ \sum_{k\geq 1} \frac{\alpha_k^2}{k^2} < \infty \bigg\},$$

endowed with the norm

$$\|v\|_{\mathfrak{U}_0}^2 = \sum_{k\geq 1} \frac{\alpha_k^2}{k^2}, \qquad v = \sum_{k\geq 1} \alpha_k e_k.$$

Note that the embedding $\mathfrak{U} \hookrightarrow \mathfrak{U}_0$ is Hilbert-Schmidt. Moreover, trajectories of W are $\mathbb{P}$-a.s. in $C([0,T]; \mathfrak{U}_0)$ (see [15]).

In the present paper, we use the brackets $\langle \cdot, \cdot \rangle$ to denote the duality between the space of distributions over $\mathbb{T}^N \times \mathbb{R}$ and $C_c^\infty(\mathbb{T}^N \times \mathbb{R})$. We denote similarly the integral

$$\langle F, G \rangle = \int_{\mathbb{T}^N} \int_{\mathbb{R}} F(x,\xi) G(x,\xi) \, \mathrm{d}x \, \mathrm{d}\xi, \qquad F \in L^p(\mathbb{T}^N \times \mathbb{R}),\ G \in L^q(\mathbb{T}^N \times \mathbb{R}),$$

where $p, q \in [1, \infty]$ are conjugate exponents. The differential operators of gradient ∇, divergence div and Laplacian Δ are always understood with respect to the space variable x.

As the next step, we introduce the kinetic formulation of (3.1) as well as the basic definitions concerning the notion of kinetic solution. The motivation for this approach is given by the nonexistence of a strong solution and, on the other hand, the nonuniqueness of weak solutions, even in simple cases. The idea is to establish an additional criterion – the kinetic formulation – which is automatically satisfied by any strong solution to (3.1) and which permits to ensure the well-posedness.

Definition 3.2.1 (Kinetic measure)**.** A mapping m from Ω to the set of nonnegative finite measures over $\mathbb{T}^N \times [0,T] \times \mathbb{R}$ is said to be a kinetic measure provided

(i) m is measurable in the following sense: for each $\psi \in C_0(\mathbb{T}^N \times [0,T] \times \mathbb{R})$ the mapping $m(\psi) : \Omega \to \mathbb{R}$ is measurable,

(ii) m vanishes for large ξ: if $B_R^c = \{\xi \in \mathbb{R};\ |\xi| \geq R\}$ then

$$\lim_{R\to\infty} \mathbb{E}\, m\big(\mathbb{T}^N \times [0,T] \times B_R^c\big) = 0, \tag{3.5}$$

(iii) for any $\psi \in C_0(\mathbb{T}^N \times \mathbb{R})$

$$\int_{\mathbb{T}^N \times [0,t] \times \mathbb{R}} \psi(x,\xi) \, \mathrm{d}m(x,s,\xi) \in L^2(\Omega \times [0,T])$$

admits a predictable representative[1].

[1]Throughout the paper, the term *representative* stands for an element of a class of equivalence.

Definition 3.2.2 (Kinetic solution)**.** Assume that, for all $p \in [1, \infty)$,

$$u \in L^p(\Omega \times [0,T], \mathcal{P}, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t; L^p(\mathbb{T}^N))$$

and

(i) there exists $C_p > 0$ such that

$$\mathbb{E} \operatorname*{ess\,sup}_{0 \le t \le T} \|u(t)\|^p_{L^p(\mathbb{T}^N)} \le C_p, \tag{3.6}$$

(ii) $\sigma \nabla u \in L^2(\Omega \times [0,T]; L^2(\mathbb{T}^N))$.

Let n_1 be a mapping from Ω to the set of nonnegative finite measures over $\mathbb{T}^N \times [0,T] \times \mathbb{R}$ defined for any Borel set $D \in \mathcal{B}(\mathbb{T}^N \times [0,T] \times \mathbb{R})$ as[2]

$$n_1(D) = \int_{\mathbb{T}^N \times [0,T]} \left[\int_{\mathbb{R}} \mathbf{1}_D(x,t,\xi)\, \mathrm{d}\delta_{u(x,t)}(\xi) \right] \big|\sigma(x)\nabla u\big|^2 \,\mathrm{d}x\,\mathrm{d}t, \quad \mathbb{P}\text{-a.s.}, \tag{3.7}$$

and let

$$f = \mathbf{1}_{u>\xi} : \Omega \times \mathbb{T}^N \times [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}.$$

Then u is said to be a kinetic solution to (3.1) with initial datum u_0 provided there exists a kinetic measure $m \ge n_1$ a.s., such that the pair $(f = \mathbf{1}_{u>\xi}, m)$ satisfies, for all $\varphi \in C_c^\infty(\mathbb{T}^N \times [0,T) \times \mathbb{R})$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
\int_0^T \big\langle f(t), \partial_t \varphi(t) \big\rangle \mathrm{d}t + \big\langle f_0, \varphi(0) \big\rangle + \int_0^T \big\langle f(t), b(\xi) \cdot \nabla\varphi(t) \big\rangle \mathrm{d}t& \\
+ \int_0^T \big\langle f(t), \operatorname{div}\big(A(x)\nabla\varphi(t)\big) \big\rangle \mathrm{d}t& \\
= -\sum_{k\ge 1} \int_0^T \int_{\mathbb{T}^N} g_k\big(x, u(x,t)\big) \varphi\big(x,t,u(x,t)\big) \mathrm{d}x\, \mathrm{d}\beta_k(t)& \\
- \frac{1}{2} \int_0^T \int_{\mathbb{T}^N} G^2\big(x, u(x,t)\big) \partial_\xi \varphi\big(x,t,u(x,t)\big) \mathrm{d}x\, \mathrm{d}t + m(\partial_\xi \varphi)&.
\end{aligned} \tag{3.8}$$

Remark 3.2.3. We emphasize that a kinetic solution is, in fact, a class of equivalence in $L^p(\Omega \times [0,T], \mathcal{P}, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t; L^p(\mathbb{T}^N))$ so not necessarily a stochastic process in the usual sense. Nevertheless, it will be seen later (see Corollary 3.3.4) that, in this class of equivalence, there exists a representative with good continuity properties, namely, $u \in C([0,T]; L^p(\mathbb{T}^N))$, $\mathbb{P}$-a.s., and therefore, it can be regarded as a stochastic process.

Remark 3.2.4. Let us also make an observation which clarifies the point (ii) in the above definition: if $u \in L^2(\Omega \times [0,T]; L^2(\mathbb{T}^N))$ then it can be shown that $\sigma \nabla u$ is well defined in $L^2(\Omega \times [0,T]; H^{-1}(\mathbb{T}^N))$ since the square-root matrix σ belongs to $W^{1,\infty}(\mathbb{T}^N)$ according to [24], [66].

By $f = \mathbf{1}_{u>\xi}$ we understand a real function of four variables, where the additional variable ξ is called velocity. In the deterministic case, i.e. corresponding to the situation $\varPhi = 0$, the equation (3.8) in the above definition is the so-called kinetic formulation of (3.1)

$$\partial_t \mathbf{1}_{u>\xi} + b(\xi) \cdot \nabla \mathbf{1}_{u>\xi} - \operatorname{div}\big(A(x)\nabla \mathbf{1}_{u>\xi}\big) = \partial_\xi m$$

[2] We will write shortly $\mathrm{d}n_1(x,t,\xi) = \big|\sigma(x)\nabla u\big|^2 \,\mathrm{d}\delta_{u(x,t)}(\xi)\,\mathrm{d}x\,\mathrm{d}t$.

where the unknown is the pair $(\mathbf{1}_{u>\xi}, m)$ and it is solved in the sense of distributions over $\mathbb{T}^N \times [0,T) \times \mathbb{R}$. In the stochastic case, we write formally[3]

$$\partial_t \mathbf{1}_{u>\xi} + b(\xi)\cdot\nabla\mathbf{1}_{u>\xi} - \operatorname{div}\big(A(x)\nabla\mathbf{1}_{u>\xi}\big) = \delta_{u=\xi}\varPhi(u)\dot{W} + \partial_\xi\Big(m - \frac{1}{2}G^2\delta_{u=\xi}\Big). \tag{3.9}$$

It will be seen later that this choice is reasonable since for any u being a strong solution to (3.1) the pair $(\mathbf{1}_{u>\xi}, n_1)$ satisfies (3.8) and consequently u is a kinetic solution to (3.1). The measure n_1 relates to the diffusion term in (3.1) and so is called parabolic dissipative measure. It gives us better regularity of solutions in the nondegeneracy zones of the diffusion matrix A which is exactly what one would expect according to the theory of (nondegenerate) parabolic SPDEs. Indeed, for the case of a nondegenerate diffusion matrix A, i.e. when the second order term defines a strongly elliptic differential operator, the kinetic solution u belongs to $L^2(\Omega; L^2(0,T; H^1(\mathbb{T}^N)))$ (cf. Definition 3.2.2(ii)). Thus, the measure $n_2 = m - n_1$ which takes account of possible singularities of solution vanishes in the nondegenerate case.

We now derive the kinetic formulation in case of a sufficiently smooth u satisfying (3.1), namely, $u \in C([0,T]; C^2(\mathbb{T}^N))$, $\mathbb{P}$-a.s.. Note, that also in this case, the measure n_2 vanishes. For almost every $x \in \mathbb{T}^N$, we aim at finding the stochastic differential of $\theta(u(x,t))$, where $\theta \in C^\infty(\mathbb{R})$ is an arbitrary test function. Such a method can be performed by the Itô formula since

$$\begin{aligned} u(x,t) = u_0(x) &- \int_0^t \operatorname{div}\big(B(u(x,s))\big)\,\mathrm{d}s + \int_0^t \operatorname{div}\big(A(x)\nabla u(x,s)\big)\,\mathrm{d}s \\ &+ \sum_{k\geq 1}\int_0^t g_k\big(x,u(x,s)\big)\,\mathrm{d}\beta_k(s), \quad \text{a.e. } (\omega,x)\in\Omega\times\mathbb{T}^N,\ \forall t\in[0,T]. \end{aligned} \tag{3.10}$$

In the following we denote by $\langle\cdot,\cdot\rangle_\xi$ the duality between the space of distributions over $\mathbb{R}$ and $C_c^\infty(\mathbb{R})$. Fix $x \in \mathbb{T}^N$ such that (3.10) holds true and consider $\mathbf{1}_{u(x,t)>\xi}$ as a (random) distribution on $\mathbb{R}$. Then

$$\langle \mathbf{1}_{u(x,t)>\xi}, \theta' \rangle_\xi = \int_{\mathbb{R}} \mathbf{1}_{u(x,t)>\xi}\theta'(\xi)\,\mathrm{d}\xi = \theta(u(x,t))$$

and the application of the Itô formula yields:

$$\begin{aligned} \mathrm{d}\langle \mathbf{1}_{u(x,t)>\xi}, \theta'\rangle_\xi = \theta'(u(x,t))\Big[&-\operatorname{div}\big(B(u(x,t))\big)\mathrm{d}t + \operatorname{div}\big(A(x)\nabla u(x,t)\big)\mathrm{d}t \\ &+ \sum_{k\geq 1} g_k(x,u(x,t))\,\mathrm{d}\beta_k(t)\Big] + \frac{1}{2}\theta''(u(x,t))G^2(u(x,t))\mathrm{d}t. \end{aligned}$$

[3]Hereafter, we employ the notation which is commonly used in papers concerning the kinetic solutions to conservation laws and write $\delta_{u=\xi}$ for the Dirac measure centered at $u(x,t)$.

Afterwards, we proceed term by term and employ the fact that all the necessary derivatives of u exists as functions

$$\begin{aligned}
\theta'(u(x,t))\,\mathrm{div}\,\big(B(u(x,t))\big) &= \theta'(u(x,t))b(u(x,t))\cdot\nabla u(x,t)\\
&= \mathrm{div}\left(\int_{-\infty}^{u(x,t)} b(\xi)\theta'(\xi)\mathrm{d}\xi\right) = \mathrm{div}\,\big(\langle b\mathbf{1}_{u(x,t)>\xi},\theta'\rangle_\xi\big)\\
\theta'(u(x,t))\,\mathrm{div}\,\big(A(x)\nabla u(x,t)\big) &= \sum_{i,j=1}^N \partial_{x_i}\big[A_{ij}(x)\theta'(u(x,t))\partial_{x_j}u(x,t)\big]\\
&\quad - \sum_{i,j=1}^N \theta''(u(x,t))\partial_{x_i}u(x,t)A_{ij}(x)\partial_{x_j}u(x,t)\\
&= \sum_{i,j=1}^N \partial_{x_i}\left(A_{ij}(x)\partial_{x_j}\int_{-\infty}^{u(x,t)}\theta'(\xi)\mathrm{d}\xi\right) + \big\langle\partial_\xi n_1(x,t),\theta'\big\rangle_\xi\\
&= \mathrm{div}\,\Big(A(x)\nabla\langle\mathbf{1}_{u(x,t)>\xi},\theta'\rangle_\xi\Big) + \big\langle\partial_\xi n_1(x,t),\theta'\big\rangle_\xi\\
\theta'(u(x,t))g_k(x,u(x,t)) &= \langle g_k(x,\xi)\delta_{u(x,t)=\xi},\theta'\rangle_\xi\\
\theta''(u(x,t))G^2(x,u(x,t)) &= \langle G^2(x,\xi)\delta_{u(x,t)=\xi},\theta''\rangle_\xi\\
&= -\big\langle\partial_\xi(G^2(x,\xi)\delta_{u(x,t)=\xi}),\theta'\big\rangle_\xi
\end{aligned}$$

Note, that according to the definition of the parabolic dissipative measure (3.7) it makes sense to write $\partial_\xi n_1(x,t)$, i.e for fixed x,t we regard $n_1(x,t)$ as a random measure on $\mathbb{R}$: for any Borel set $D_1\in\mathcal{B}(\mathbb{R})$

$$n_1(x,t,D_1) = \big|\sigma(x)\nabla u(x,t)\big|^2\delta_{u(x,t)}(D_1),\qquad \mathbb{P}\text{-a.s..}$$

In the following, we distinguish between two situations. In the first case, we intend to use test functions independent on t. We set $\theta(\xi)=\int_{-\infty}^{\xi}\varphi_1(\zeta)\,\mathrm{d}\zeta$ for some test function $\varphi_1\in C_c^\infty(\mathbb{R})$ and test the above against $\varphi_2\in C^\infty(\mathbb{T}^N)$. Since linear combinations of the test functions $\psi(x,\xi)=\varphi_1(\xi)\varphi_2(x)$ form a dense subset of $C_c^\infty(\mathbb{T}^N\times\mathbb{R})$ we obtain for any $\psi\in C_c^\infty(\mathbb{T}^N\times\mathbb{R})$, $t\in[0,T]$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\big\langle f(t),\psi\big\rangle - \big\langle f_0,\psi\big\rangle - \int_0^t\big\langle f(s),b(\xi)\cdot\nabla\psi\big\rangle\,\mathrm{d}s - \int_0^t\big\langle f(s),\mathrm{div}\,\big(A(x)\nabla\psi\big)\big\rangle\,\mathrm{d}s\\
&= \int_0^t\big\langle\delta_{u=\xi}\,\varPhi(u)\,\mathrm{d}W,\psi\big\rangle + \frac{1}{2}\int_0^t\big\langle\delta_{u=\xi}\,G^2,\partial_\xi\psi\big\rangle\,\mathrm{d}s - \big\langle n_1,\partial_\xi\psi\big\rangle([0,t)),
\end{aligned}$$

where $\big\langle n_1,\partial_\xi\psi\big\rangle([0,t)) = n_1\big(\partial_\xi\psi\mathbf{1}_{[0,t)}\big)$. In order to allow test functions from $C_c^\infty(\mathbb{T}^N\times[0,T)\times\mathbb{R})$, take $\varphi_3\in C_c^\infty([0,T))$ and apply the Itô formula to calculate the stochastic differential of the product $\langle f(t),\psi\rangle\varphi_3(t)$. We have, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\big\langle f(t),\psi\big\rangle\varphi_3(t) - \big\langle f_0,\psi\big\rangle\varphi_3(0) - \int_0^t\big\langle f(s),b(\xi)\cdot\nabla\psi\big\rangle\varphi_3(s)\,\mathrm{d}s\\
&\quad - \int_0^t\big\langle f(s),\mathrm{div}\,\big(A(x)\nabla\psi\big)\big\rangle\varphi_3(s)\,\mathrm{d}s\\
&= \int_0^t\big\langle\delta_{u=\xi}\,\varPhi(u)\varphi_3(s)\,\mathrm{d}W,\psi\big\rangle + \frac{1}{2}\int_0^t\big\langle\delta_{u=\xi}\,G^2,\partial_\xi\psi\big\rangle\varphi_3(s)\,\mathrm{d}s\\
&\quad - n_1\big(\partial_\xi\psi\mathbf{1}_{[0,t)}\varphi_3\big) + \int_0^t\big\langle f(s),\psi\big\rangle\partial_s\varphi_3(s)\,\mathrm{d}s.
\end{aligned}$$

Evaluating this process at $t=T$ and setting $\varphi(x,t,\xi)=\psi(x,\xi)\varphi_3(t)$ yields the equation (3.8) hence $f=\mathbf{1}_{u>\xi}$ is a distributional solution to the kinetic formulation (3.9) with $n_2=0$. Therefore any strong solution of (3.1) is a kinetic solution in the sense of Definition 3.2.2.

Concerning the point (ii) in Definition 3.2.2, it was already mentioned in Remark 3.2.4 that $\sigma\,\nabla u$ is well defined in $L^2(\Omega\times[0,T];H^{-1}(\mathbb{T}^N))$. As we assume more in Definition 3.2.2(ii) we obtain the following chain rule formula, which will be used in the proof of Theorem 3.3.3,

$$\sigma\,\nabla f=\sigma\,\nabla u\,\delta_{u=\xi}\quad\text{in}\quad\mathcal{D}'(\mathbb{T}^N\times\mathbb{R}),\ \text{a.e.}\ (\omega,t)\in\Omega\times[0,T].\tag{3.11}$$

It is a consequence of the next result.

Lemma 3.2.5. *Assume that $v\in L^2(\mathbb{T}^N)$ and $\sigma\,(\nabla v)\in L^2(\mathbb{T}^N)$. If $g=\mathbf{1}_{v>\xi}$ then it holds true*

$$\sigma\,\nabla g=\sigma\,\nabla v\,\delta_{v=\xi}\qquad\textit{in}\qquad\mathcal{D}'(\mathbb{T}^N\times\mathbb{R}).$$

Proof. In order to prove this claim, we denote by σ^i the i^{th} row of σ. Let us fix test functions $\psi_1\in C^\infty(\mathbb{T}^N)$, $\psi_2\in C_c^\infty(\mathbb{R})$ and define $\theta(\xi)=\int_{-\infty}^{\xi}\psi_2(\zeta)\,\mathrm{d}\zeta$. We denote by $\langle\cdot,\cdot\rangle_x$ the duality between the space of distributions over $\mathbb{T}^N$ and $C^\infty(\mathbb{T}^N)$. It holds

$$\begin{aligned}\big\langle\sigma^i\,\nabla g,\psi_1\psi_2\big\rangle&=-\Big\langle\operatorname{div}(\sigma^i\psi_1),\int_{-\infty}^{v}\psi_2(\xi)\,\mathrm{d}\xi\Big\rangle_x=-\big\langle\operatorname{div}\big(\sigma^i\psi_1\big),\theta(v)\big\rangle_x\\&=\big\langle\sigma^i\,\nabla\theta(v),\psi_1\big\rangle_x.\end{aligned}$$

If the following was true

$$\sigma^i\,\nabla\theta(v)=\theta'(v)\,\sigma^i\,\nabla v\qquad\text{in}\qquad\mathcal{D}'(\mathbb{T}^N),\tag{3.12}$$

we would obtain

$$\big\langle\sigma^i\,\nabla g,\psi_1\psi_2\big\rangle=\big\langle\theta'(v)\,\sigma^i\,\nabla v,\psi_1\big\rangle_x=\big\langle\sigma^i\,\nabla v\,\delta_{v=\xi},\psi_1\psi_2\big\rangle$$

and the proof would be complete.

Hence it remains to verify (3.12). Towards this end, let us consider an approximation to the identity on $\mathbb{T}^N$, denoted by (ϱ_τ). To be more precise, let $\tilde\varrho\in C_c^\infty(\mathbb{R}^N)$ be nonnegative symmetric function satisfying $\int_{\mathbb{R}^N}\tilde\varrho=1$ and $\operatorname{supp}\tilde\varrho\subset B(0,1/2)$. This function can be easily extended to become $\mathbb{Z}^N$-periodic, let this modification denote by $\bar\varrho$. Now it is correct to define $\varrho=\bar\varrho\circ q^{-1}$, where q denotes the quotient mapping $q:\mathbb{R}^N\to\mathbb{T}^N=\mathbb{R}^N/\mathbb{Z}^N$, and finally

$$\varrho_\tau(x)=\frac{1}{\tau^N}\,\varrho\Big(\frac{x}{\tau}\Big).$$

Since the identity (3.12) is fulfilled by any sufficiently regular v, let us consider v^τ, the mollifications of v given by (ϱ_τ). We have

$$\sigma^i\,\nabla\theta(v^\tau)\longrightarrow\sigma^i\,\nabla\theta(v)\qquad\text{in}\qquad\mathcal{D}'(\mathbb{T}^N).$$

In order to obtain convergence of the corresponding right hand sides, i.e.

$$\theta'(v^\tau)\,\sigma^i\,\nabla v^\tau\longrightarrow\theta'(v)\,\sigma^i\,\nabla v\qquad\text{in}\qquad\mathcal{D}'(\mathbb{T}^N),$$

we employ similar arguments as in the commutation lemma of DiPerna and Lions (see [17, Lemma II.1]). Namely, since $\sigma^i\,(\nabla v)\in L^2(\mathbb{T}^N)$ it is approximated in $L^2(\mathbb{T}^N)$ by its mollifications $[\sigma^i\,\nabla v]^\tau$. Consequently,

$$\theta'(v^\tau)\big[\sigma^i\,\nabla v\big]^\tau\longrightarrow\theta'(v)\,\sigma^i\,\nabla v\qquad\text{in}\qquad\mathcal{D}'(\mathbb{T}^N).$$

Thus, it is enough to show that

$$\theta'(v^\tau)\Big(\sigma^i\,\nabla v^\tau-\big[\sigma^i\,\nabla v\big]^\tau\Big)\longrightarrow 0\qquad\text{in}\qquad\mathcal{D}'(\mathbb{T}^N).\tag{3.13}$$

It holds

$$\begin{aligned}&\sigma^i(x)\nabla v^\tau(x)-\big[\sigma^i\,\nabla v\big]^\tau(x)\\&=\int_{\mathbb{T}^N}v(y)\,\sigma^i(x)(\nabla\varrho_\tau)(x-y)\,\mathrm{d}y+\int_{\mathbb{T}^N}v(y)\,\mathrm{div}_y\,\big(\sigma^i(y)\varrho_\tau(x-y)\big)\,\mathrm{d}y\\&=-\int_{\mathbb{T}^N}v(y)\big(\sigma^i(y)-\sigma^i(x)\big)(\nabla\varrho_\tau)(x-y)\mathrm{d}y+\int_{\mathbb{T}^N}v(y)\,\mathrm{div}\,\big(\sigma^i(y)\big)\varrho_\tau(x-y)\mathrm{d}y.\end{aligned}$$

The second term on the right hand side is the mollification of $v\,\mathrm{div}\,\sigma^i\in L^2(\mathbb{T}^N)$ hence converges in $L^2(\mathbb{T}^N)$ to $v\,\mathrm{div}\,\sigma^i$. We will show that the first term converges in $L^1(\mathbb{T}^N)$ to $-v\,\mathrm{div}\,\sigma^i$. Since $\tau|\nabla\varrho_\tau|(\cdot)\leq C\varrho_{2\tau}(\cdot)$ with a constant independent on τ, we obtain the following estimate

$$\bigg\|\int_{\mathbb{T}^N}v(y)\big(\sigma^i(y)-\sigma^i(x)\big)(\nabla\varrho_\tau)(x-y)\,\mathrm{d}y\bigg\|_{L^2(\mathbb{T}^N)}\leq C\|\sigma^i\|_{W^{1,\infty}(\mathbb{T}^N)}\|v\|_{L^2(\mathbb{T}^N)}.$$

Due to this estimate, it is sufficient to consider v and σ^i smooth and the general case can be concluded by a density argument. We infer[4]

$$\begin{aligned}&-\int_{\mathbb{T}^N}v(y)\big(\sigma^i(y)-\sigma^i(x)\big)(\nabla\varrho_\tau)(x-y)\,\mathrm{d}y\\&\qquad=-\frac{1}{\tau^{N+1}}\int_{\mathbb{T}^N}\int_0^1 v(y)\,\mathrm{D}\sigma^i\big(x+r(y-x)\big)(y-x)\cdot(\nabla\varrho)\Big(\frac{x-y}{\tau}\Big)\,\mathrm{d}r\,\mathrm{d}y\\&\qquad=\int_{\mathbb{T}^N}\int_0^1 v(x-\tau z)\,\mathrm{D}\sigma^i(x-r\tau z)z\cdot(\nabla\varrho)(z)\,\mathrm{d}r\,\mathrm{d}z\\&\longrightarrow v(x)\,\mathrm{D}\sigma^i(x):\int_{\mathbb{T}^N}z\otimes(\nabla\varrho)(z)\,\mathrm{d}z,\qquad\forall x\in\mathbb{T}^N.\end{aligned}$$

Integration by parts now yields

$$\int_{\mathbb{T}^N}z\otimes(\nabla\varrho)(z)\,\mathrm{d}z=-\mathrm{Id}\tag{3.14}$$

hence

$$v(x)\,\mathrm{D}\sigma^i(x):\int_{\mathbb{T}^N}z\otimes(\nabla\varrho)(z)\,\mathrm{d}z=-v(x)\,\mathrm{div}\,\big(\sigma^i(x)\big),\qquad\forall x\in\mathbb{T}^N,$$

and the convergence in $L^1(\mathbb{T}^N)$ follows by the Vitali convergence theorem from the above estimate. Employing the Vitali convergence theorem again, we obtain (3.13) and consequently also (3.12) which completes the proof. □

[4] By : we denote the component-wise inner product of matrices and by $\otimes$ the tensor product.

We proceed by two related definitions, which will be useful especially in the proof of uniqueness.

Definition 3.2.6 (Young measure)**.** Let (X,λ) be a finite measure space. A mapping ν from X to the set of probability measures on $\mathbb{R}$ is said to be a Young measure if, for all $\psi\in C_b(\mathbb{R})$, the map $z\mapsto\nu_z(\psi)$ from X into $\mathbb{R}$ is measurable. We say that a Young measure ν vanishes at infinity if, for all $p\geq 1$,

$$\int_X\int_{\mathbb{R}}|\xi|^p\mathrm{d}\nu_z(\xi)\,\mathrm{d}\lambda(z)<\infty.$$

Definition 3.2.7 (Kinetic function)**.** Let (X,λ) be a finite measure space. A measurable function $f:X\times\mathbb{R}\to[0,1]$ is said to be a kinetic function if there exists a Young measure ν on X vanishing at infinity such that, for λ-a.e. $z\in X$, for all $\xi\in\mathbb{R}$,

$$f(z,\xi)=\nu_z(\xi,\infty).$$

Remark 3.2.8. Note, that if f is a kinetic function then $\partial_\xi f=-\nu$ for λ-a.e. $z\in X$. Similarly, let u be a kinetic solution of (3.1) and consider $f=\mathbf{1}_{u>\xi}$. We have $\partial_\xi f=-\delta_{u=\xi}$, where $\nu=\delta_{u=\xi}$ is a Young measure on $\Omega\times\mathbb{T}^N\times[0,T]$. Therefore, the expression (3.8) can be rewritten in the following form: for all $\varphi\in C_c^\infty(\mathbb{T}^N\times[0,T)\times\mathbb{R})$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\int_0^T\big\langle f(t),\partial_t\varphi(t)\big\rangle\mathrm{d}t+\big\langle f_0,\varphi(0)\big\rangle+\int_0^T\big\langle f(t),b(\xi)\cdot\nabla\varphi(t)\big\rangle\mathrm{d}t&\\ +\int_0^T\big\langle f(t),\mathrm{div}\,\big(A(x)\nabla\varphi(t)\big)\big\rangle\mathrm{d}t&\\ =-\sum_{k\geq1}\int_0^T\int_{\mathbb{T}^N}\int_{\mathbb{R}}g_k(x,\xi)\varphi(x,t,\xi)\mathrm{d}\nu_{x,t}(\xi)\,\mathrm{d}x\,\mathrm{d}\beta_k(t)&\\ -\frac12\int_0^T\int_{\mathbb{T}^N}\int_{\mathbb{R}}G^2(x,\xi)\partial_\xi\varphi(x,t,\xi)\mathrm{d}\nu_{x,t}(\xi)\,\mathrm{d}x\,\mathrm{d}t+m(\partial_\xi\varphi).&\end{aligned}\tag{3.15}$$

For a general kinetic function f with corresponding Young measure ν, the above formulation leads to the notion of generalized kinetic solution as used in [16]. Although this concept is not established here, the notation will be used throughout the paper, i.e. we will often write $\nu_{x,t}(\xi)$ instead of $\delta_{u(x,t)=\xi}$.

Lemma 3.2.9. *Let (X,λ) be a finite measure space such that $L^1(X)$ is separable.*[5] *Let $\{f_n;\,n\in\mathbb{N}\}$ be a sequence of kinetic functions on $X\times\mathbb{R}$, i.e. $f_n(z,\xi)=\nu_z^n(\xi,\infty)$ where ν^n are Young measures on X. Suppose that, for some $p\geq1$,*

$$\sup_{n\in\mathbb{N}}\int_X\int_{\mathbb{R}}|\xi|^p\mathrm{d}\nu_z^n(\xi)\,\mathrm{d}\lambda(z)<\infty.$$

Then there exists a kinetic function f on $X\times\mathbb{R}$ and a subsequence still denoted by $\{f_n;\,n\in\mathbb{N}\}$ such that

$$f_n\xrightarrow{w^*}f,\qquad\textit{in}\qquad L^\infty(X\times\mathbb{R})\textit{-weak}^*.$$

Proof. The proof can be found in [16, Corollary 6]. □

[5] According to [14, Proposition 3.4.5], it is sufficient to assume that the corresponding σ-algebra is countably generated.

To conclude this section we state the main result of the paper.

Theorem 3.2.10. *Let $u_0 \in L^p(\Omega; L^p(\mathbb{T}^N))$, for all $p \in [1,\infty)$. Under the above assumptions, there exists a unique kinetic solution to the problem (3.1) and it has almost surely continuous trajectories in $L^p(\mathbb{T}^N)$, for all $p \in [1,\infty)$. Moreover, if u_1, u_2 are kinetic solutions to (3.1) with initial data $u_{1,0}$ and $u_{2,0}$, respectively, then for all $t \in [0,T]$*

$$\mathbb{E}\|u_1(t)-u_2(t)\|_{L^1(\mathbb{T}^N)} \leq \mathbb{E}\|u_{1,0}-u_{2,0}\|_{L^1(\mathbb{T}^N)}.$$

3.3 Uniqueness

We begin with the question of uniqueness. Due to the following proposition, we obtain an auxiliary property of kinetic solutions, which will be useful later on in the proof of the comparison principle in Theorem 3.3.3.

Proposition 3.3.1 (Left- and right-continuous representatives)**.** *Let u be a kinetic solution to (3.1). Then $f = \mathbf{1}_{u>\xi}$ admits representatives f^- and f^+ which are almost surely left- and right-continuous, respectively, at all points $t^* \in [0,T]$ in the sense of distributions over $\mathbb{T}^N \times \mathbb{R}$. More precisely, for all $t^* \in [0,T]$ there exist kinetic functions $f^{*,\pm}$ on $\Omega \times \mathbb{T}^N \times \mathbb{R}$ such that setting $f^\pm(t^*) = f^{*,\pm}$ yields $f^\pm = f$ almost everywhere and*

$$\big\langle f^\pm(t^* \pm \varepsilon), \psi\big\rangle \longrightarrow \big\langle f^\pm(t^*), \psi\big\rangle \quad \varepsilon \downarrow 0 \quad \forall \psi \in C_c^2(\mathbb{T}^N \times \mathbb{R}) \quad \mathbb{P}\text{-a.s.}.$$

Moreover, $f^+ = f^-$ for all $t^* \in [0,T]$ except for some at most countable set.

Proof. Let $\psi \in C_c^2(\mathbb{T}^N \times \mathbb{R})$ and $\alpha \in C_c^1([0,T))$ and set $\varphi(x,t,\xi) = \psi(x,\xi)\alpha(t)$. Integration by parts and the stochastic version of Fubini's theorem applied to (3.15) yield

$$\int_0^T g_\psi(t)\alpha'(t)\mathrm{d}t + \langle f_0, \psi\rangle\alpha(0) = \langle m, \partial_\xi\psi\rangle(\alpha) \qquad \mathbb{P}\text{-a.s.}$$

where

$$\begin{aligned} g_\psi(t) = \big\langle f(t), \psi\big\rangle &- \int_0^t \big\langle f(s), b(\xi)\cdot\nabla\psi\big\rangle \mathrm{d}s - \int_0^t \big\langle f(t), \operatorname{div}\big(A(x)\nabla\psi\big)\big\rangle\,\mathrm{d}s \\ &- \sum_{k\geq 1}\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}} g_k(x,\xi)\psi(x,\xi)\,\mathrm{d}\nu_{x,s}(\xi)\,\mathrm{d}x\,\mathrm{d}\beta_k(s) \\ &- \frac{1}{2}\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}} \partial_\xi\psi(x,\xi)G^2(x,\xi)\,\mathrm{d}\nu_{x,s}(\xi)\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{3.16}$$

Hence $\partial_t g_\psi$ is a (pathwise) Radon measure on $[0,T]$ and by the Riesz representation theorem $g_\psi \in BV([0,T])$. Moreover, apart from the first one all terms in (3.16) are continuous in t hence $\langle f,\psi\rangle \in BV([0,T])$ almost surely. Let us denote the corresponding set of full measure $\Omega_\psi \subset \Omega$ to indicate its dependence on the chosen test function. Due to the properties of BV-functions [4, Theorem 3.28], we obtain that $\langle f(\cdot,\omega),\psi\rangle$, $\omega \in \Omega_\psi$, admits left- and right-continuous representatives which coincide except for at most countable set. Let them be denoted by $\langle f(\cdot,\omega),\psi\rangle^\pm$.

We will now establish three convergence facts (3.17), (3.18), (3.20) which together imply the statement of the proposition. On the one hand, we infer for all $t^* \in [0,T)$ and $\omega \in \Omega_\psi$ that

$$\big\langle f(t^*,\omega),\psi\big\rangle^+ = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t^*}^{t^*+\varepsilon}\big\langle f(t,\omega),\psi\big\rangle^+\,\mathrm{d}t = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t^*}^{t^*+\varepsilon}\big\langle f(t,\omega),\psi\big\rangle\,\mathrm{d}t$$

hence by the dominated convergence theorem

$$\frac{1}{\varepsilon}\int_{t^*}^{t^*+\varepsilon}\big\langle f(t),\psi\big\rangle\,\mathrm{d}t\overset{w^*}{\longrightarrow}\big\langle f(t^*),\psi\big\rangle^+\qquad\text{in}\qquad L^\infty(\Omega)\text{-}w^*. \tag{3.17}$$

On the other hand, we consider a sequence $\varepsilon_n\downarrow 0$ and define (independently on ψ) for almost all $\omega\in\Omega$, $x\in\mathbb{T}^N$, $\xi\in\mathbb{R}$, and all $t^*\in[0,T)$,

$$f_n(x,t^*,\xi):=\frac{1}{\varepsilon_n}\int_{t^*}^{t^*+\varepsilon_n}f(x,t,\xi)\,\mathrm{d}t.$$

For any $t^*\in[0,T)$ the function $f_n(t^*)$ is a kinetic function on $X=\Omega\times\mathbb{T}^N$ and by (3.6) the assumptions of Lemma 3.2.9 are fulfilled. Accordingly, there exists a kinetic function $f^{*,+}$ and a subsequence (n_k^*) (which also depends on t^*) such that

$$f_{n_k^*}(t^*)\overset{w^*}{\longrightarrow}f^{*,+}\qquad\text{in}\qquad L^\infty(\Omega\times\mathbb{T}^N\times\mathbb{R})\text{-}w^*. \tag{3.18}$$

Note, that the domain of definition of $f^{*,+}$ does not depend on ψ. As a consequence, we obtain by the Fubini theorem, (3.17) and (3.18) that there exists a set of full probability, denoted by $\tilde\Omega_\psi$, such that

$$\big\langle f^{*,+}(\omega),\psi\big\rangle=\big\langle f(t^*,\omega),\psi\big\rangle^+\qquad\forall\omega\in\tilde\Omega_\psi. \tag{3.19}$$

Thus, the limit in (3.18) is independent of the chosen sequence (ε_n) and subsequence $(\varepsilon_{n_k^*})$. And furthermore, due to the Lebesgue differentiation theorem,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t^*}^{t^*+\varepsilon}f(\omega,x,t,\xi)\,\mathrm{d}t=f(\omega,x,t^*,\xi)\quad\text{for a.e. }(\omega,x,t^*,\xi)$$

hence by the dominated convergence theorem, for almost every $t^*\in[0,T)$,

$$\frac{1}{\varepsilon}\int_{t^*}^{t^*+\varepsilon}f(t)\,\mathrm{d}t\overset{w^*}{\longrightarrow}f(t^*)\qquad\text{in}\qquad L^\infty(\Omega\times\mathbb{T}^N\times\mathbb{R})\text{-}w^*. \tag{3.20}$$

As a consequence, it follows from (3.18) that $f^{*,+}(\omega,x,\xi)=f(\omega,x,t^*,\xi)$ almost everywhere in ω,x,ξ and the exceptional set here does not depend on ψ.

Altogether, setting $f^+(t^*)=f^{*,+}$, $t^*\in[0,T)$, we finally conclude that $f^+=f$ almost everywhere in ω,x,t,ξ and that $\langle f^+,\psi\rangle$ is right-continuous, i.e. for all $t^*\in[0,T)$

$$\big\langle f^+(t^*+\varepsilon,\omega),\psi\big\rangle\longrightarrow\big\langle f^+(t^*,\omega),\psi\big\rangle\qquad\forall\omega\in\tilde\Omega_\psi.$$

The rest follows immediately from the fact that the space $C_c^2(\mathbb{T}^N\times\mathbb{R})$ (endowed with the topology of the uniform convergence on any compact set of functions and their first and second derivatives) is separable. Indeed, if $\mathcal{D}_1$ is a countable dense subset of $C_c^2(\mathbb{T}^N\times\mathbb{R})$, then there exists a set of full probability $\Omega_{\mathcal{D}_1}\subset\Omega$ such that

$$\big\langle f^+(t^*+\varepsilon,\omega),\psi\big\rangle\longrightarrow\big\langle f^+(t^*,\omega),\psi\big\rangle\quad\varepsilon\downarrow 0\quad\forall\psi\in\mathcal{D}_1\quad\forall\omega\in\tilde\Omega_{\mathcal{D}_1}.$$

Let now $\psi\in C^2_c(\mathbb{T}^N\times\mathbb{R})$ be arbitrary. There exists $(\psi_n)\subset\mathcal{D}_1$ so that $\psi_n\rightarrow\psi$ in $C^2_c(\mathbb{T}^N\times\mathbb{R})$ and

$$\begin{aligned}\big|\big\langle f^+(t^*+\varepsilon),\psi\big\rangle-\big\langle f^+(t^*),\psi\big\rangle\big|\leq&\big|\big\langle f^+(t^*+\varepsilon),\psi-\psi_n\big\rangle\big|\\&+\big|\big\langle f^+(t^*+\varepsilon),\psi_n\big\rangle-\big\langle f^+(t^*),\psi_n\big\rangle\big|\\&+\big|\big\langle f^+(t^*),\psi-\psi_n\big\rangle\big|.\end{aligned}$$

If we restrict ourselves on $\Omega_{\mathcal{D}_1}$ then the latter converge to zero, due to the boundedness of f^+, and the claim follows. The proof of existence of the left-continuous representative f^- can be carried out similarly and so will be left to the reader.

Now, it only remains to verify that there exists a countable set $I\subset(0,T)$ such that $f^+(t^*)$ and $f^-(t^*)$, equivalence classes in $L^\infty(\Omega\times\mathbb{T}^N\times\mathbb{R})$, coincide for all $t^*\in(0,T)\setminus I$. Due to separability of the test function space, it is enough to show that

$$\big\langle f^+(t^*),\psi\big\rangle=\big\langle f^-(t^*),\psi\big\rangle\quad\forall\psi\in\mathcal{D}_1\quad\mathbb{P}\text{-a.s.}\quad\forall t^*\in(0,T)\setminus I.$$

However, it follows directly from (3.19) and the fact that

$$\big\langle f(t^*,\omega),\psi\big\rangle^+=\big\langle f(t^*,\omega),\psi\big\rangle^-\quad\forall\omega\in\Omega_\psi$$

for all $t^*\in(0,T)$ except for an at most countable set. □

From now on, we will work with these two fixed representatives of f and we can take any of them in an integral with respect to time or in a stochastic integral.

As the next step towards the proof of the comparison principle, we need a technical proposition relating two kinetic solutions of (3.1). We will also use the following notation: if $f:X\times\mathbb{R}\rightarrow[0,1]$ is a kinetic function, we denote by $\bar f$ the conjugate function $\bar f=1-f$.

Proposition 3.3.2. *Let u_1, u_2 be two kinetic solutions to (3.1) and denote $f_1=\mathbf{1}_{u_1>\xi}$, $f_2=\mathbf{1}_{u_2>\xi}$. Then for $t\in[0,T]$ and any nonnegative functions $\varrho\in C^\infty(\mathbb{T}^N)$, $\psi\in C^\infty_c(\mathbb{R})$ we have*

$$\begin{aligned}&\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho(x-y)\psi(\xi-\zeta)f_1^\pm(x,t,\xi)\bar f_2^\pm(y,t,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\\&\leq\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho(x-y)\psi(\xi-\zeta)f_{1,0}(x,\xi)\bar f_{2,0}(y,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y+\mathrm{I}+\mathrm{J}+\mathrm{K},\end{aligned}\tag{3.21}$$

where

$$\mathrm{I}=\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_1\bar f_2\big(b(\xi)-b(\zeta)\big)\cdot\nabla_x\alpha(x,\xi,y,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s,$$

$$\begin{aligned}\mathrm{J}=&\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_1\bar f_2\sum_{i,j=1}^N\partial_{y_j}\big(A_{ij}(y)\partial_{y_i}\alpha\big)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\&+\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}f_1\bar f_2\sum_{i,j=1}^N\partial_{x_j}\big(A_{ij}(x)\partial_{x_i}\alpha\big)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\&-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\alpha(x,\xi,y,\zeta)\,\mathrm{d}\nu^1_{x,s}(\xi)\,\mathrm{d}x\,\mathrm{d}n_{2,1}(y,s,\zeta)\\&-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\alpha(x,\xi,y,\zeta)\,\mathrm{d}\nu^2_{y,s}(\zeta)\,\mathrm{d}y\,\mathrm{d}n_{1,1}(x,s,\xi),\end{aligned}$$

$$\mathrm{K}=\frac{1}{2}\mathbb{E}\int_0^t\!\!\int_{(\mathbb{T}^N)^2}\!\int_{\mathbb{R}^2}\alpha(x,\xi,y,\zeta)\sum_{k\geq 1}\big|g_k(x,\xi)-g_k(y,\zeta)\big|^2\mathrm{d}\nu^1_{x,s}(\xi)\mathrm{d}\nu^2_{y,s}(\zeta)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s,$$

and the function α is defined as $\alpha(x,\xi,y,\zeta)=\varrho(x-y)\psi(\xi-\zeta)$.

Proof. Let us denote by $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ the scalar product in $L^2(\mathbb{T}^N_x\times\mathbb{T}^N_y\times\mathbb{R}_\xi\times\mathbb{R}_\zeta)$. In order to prove the statement in the case of f_1^+, $\bar f_2^+$, we employ similar calculations as in [16, Proposition 9] to obtain

$$\begin{aligned}
\mathbb{E}\langle\!\langle f_1^+(t)\bar f_2^+(t),\alpha\rangle\!\rangle &= \mathbb{E}\langle\!\langle f_{1,0}\bar f_{2,0},\alpha\rangle\!\rangle\\
&\quad+\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_1\bar f_2\big(b(\xi)-b(\zeta)\big)\cdot\nabla_x\alpha\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\quad+\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_1\bar f_2\sum_{i,j=1}^N\partial_{y_j}\big(A_{ij}(y)\partial_{y_i}\alpha\big)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\quad+\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_1\bar f_2\sum_{i,j=1}^N\partial_{x_j}\big(A_{ij}(x)\partial_{x_i}\alpha\big)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\quad+\frac{1}{2}\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\bar f_2\partial_\xi\alpha\,G_1^2\,\mathrm{d}\nu^1_{x,s}(\xi)\,\mathrm{d}\zeta\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s\\
&\quad-\frac{1}{2}\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_1\partial_\zeta\alpha\,G_2^2\,\mathrm{d}\nu^2_{y,s}(\zeta)\,\mathrm{d}\xi\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s\\
&\quad-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} G_{1,2}\alpha\,\mathrm{d}\nu^1_{x,s}(\xi)\,\mathrm{d}\nu^2_{y,s}(\zeta)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\quad-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\bar f_2^-\partial_\xi\alpha\,\mathrm{d}m_1(x,s,\xi)\,\mathrm{d}\zeta\,\mathrm{d}y\\
&\quad+\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_1^+\partial_\zeta\alpha\,\mathrm{d}m_2(y,s,\zeta)\,\mathrm{d}\xi\,\mathrm{d}x.
\end{aligned}\tag{3.22}$$

In particular, since $\alpha\geq 0$, the last term in (3.22) satisfies

$$\begin{aligned}
&\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_1^+\partial_\zeta\alpha\,\mathrm{d}m_2(y,s,\zeta)\,\mathrm{d}\xi\,\mathrm{d}x\\
&\qquad=-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\alpha\,\mathrm{d}\nu^1_{x,s}(\xi)\,\mathrm{d}x\,\mathrm{d}n_{2,1}(y,s,\zeta)\\
&\qquad\quad-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\alpha\,\mathrm{d}\nu^1_{x,s}(\xi)\,\mathrm{d}x\,\mathrm{d}n_{2,2}(y,s,\zeta)\\
&\qquad\leq-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\alpha\,\mathrm{d}\nu^1_{x,s}(\xi)\,\mathrm{d}x\,\mathrm{d}n_{2,1}(y,s,\zeta)
\end{aligned}$$

and by symmetry

$$\begin{aligned}
&-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\bar f_2^-\partial_\xi\alpha\,\mathrm{d}m_1(x,s,\xi)\,\mathrm{d}\zeta\,\mathrm{d}y\\
&\qquad\leq-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\alpha\,\mathrm{d}\nu^2_{y,s}(\zeta)\,\mathrm{d}y\,\mathrm{d}n_{1,1}(x,s,\xi).
\end{aligned}$$

Thus, the desired estimate (3.21) follows.

In the case of f_1^-, $\bar{f}_2^-$ we take $t_n \uparrow t$, write (3.21) for $f_1^+(t_n)$, $\bar{f}_2^+(t_n)$ and let $n \to \infty$. $\square$

Theorem 3.3.3 (Comparison principle)**.** *Let u be a kinetic solution to* (3.1)*. Then there exist u^+ and u^-, representatives of u, such that, for all $t \in [0,T]$, $f^\pm(x,t,\xi) = \mathbf{1}_{u^\pm(x,t)>\xi}$ for a.e. (ω, x, ξ). Moreover, if u_1, u_2 are kinetic solutions to* (3.1) *with initial data $u_{1,0}$ and $u_{2,0}$, respectively, then for all $t \in [0,T]$*

$$\mathbb{E}\|u_1^\pm(t) - u_2^\pm(t)\|_{L^1(\mathbb{T}^N)} \le \mathbb{E}\|u_{1,0} - u_{2,0}\|_{L^1(\mathbb{T}^N)}. \tag{3.23}$$

Proof. Denote $f_1 = \mathbf{1}_{u_1>\xi}$, $f_2 = \mathbf{1}_{u_2>\xi}$. Let (ψ_δ), (ϱ_τ) be approximations to the identity on $\mathbb{R}$ and $\mathbb{T}^N$, respectively. Namely, let $\psi \in C_c^\infty(\mathbb{R})$ be a nonnegative symmetric function satisfying $\int_\mathbb{R} \psi = 1$, $\operatorname{supp}\psi \subset (-1,1)$ and set

$$\psi_\delta(\xi) = \frac{1}{\delta}\,\psi\Big(\frac{\xi}{\delta}\Big).$$

For the space variable $x \in \mathbb{T}^N$, we employ the approximation to the identity defined in Lemma 3.2.5. Then we have

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{T}^N}\int_\mathbb{R} f_1^\pm(x,t,\xi)\bar{f}_2^\pm(x,t,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\\
&\quad = \mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \varrho_\tau(x-y)\psi_\delta(\xi-\zeta) f_1^\pm(x,t,\xi)\bar{f}_2^\pm(y,t,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y + \eta_t(\tau,\delta),
\end{aligned}$$

where $\lim_{\tau,\delta\to 0}\eta_t(\tau,\delta) = 0$. With regard to Proposition 3.3.2, we need to find suitable bounds for terms I, J, K.

Since b has at most polynomial growth, there exist $C > 0$, $p > 1$ such that

$$\big|b(\xi) - b(\zeta)\big| \le \varGamma(\xi,\zeta)|\xi - \zeta|, \qquad \varGamma(\xi,\zeta) \le C\big(1 + |\xi|^{p-1} + |\zeta|^{p-1}\big).$$

Hence

$$|\mathrm{I}| \le \mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_1\bar{f}_2\varGamma(\xi,\zeta)|\xi-\zeta|\psi_\delta(\xi-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\big|\nabla_x\varrho_\tau(x-y)\big|\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.$$

As the next step we apply integration by parts with respect to ζ, ξ. Focusing only on the relevant integrals we get

$$\begin{aligned}
&\int_\mathbb{R} f_1(\xi)\int_\mathbb{R} \bar{f}_2(\zeta)\varGamma(\xi,\zeta)|\xi-\zeta|\psi_\delta(\xi-\zeta)\mathrm{d}\zeta\,\mathrm{d}\xi\\
&\qquad = \int_\mathbb{R} f_1(\xi)\int_\mathbb{R} \varGamma(\xi,\zeta')|\xi-\zeta'|\psi_\delta(\xi-\zeta')\mathrm{d}\zeta'\,\mathrm{d}\xi\\
&\qquad\quad - \int_{\mathbb{R}^2} f_1(\xi)\int_{-\infty}^{\zeta} \varGamma(\xi,\zeta')|\xi-\zeta'|\psi_\delta(\xi-\zeta')\mathrm{d}\zeta'\,\mathrm{d}\xi\,\mathrm{d}\nu_{y,s}^2(\zeta)\\
&\qquad = \int_{\mathbb{R}^2} f_1(\xi)\int_{\zeta}^{\infty} \varGamma(\xi,\zeta')|\xi-\zeta'|\psi_\delta(\xi-\zeta')\mathrm{d}\zeta'\,\mathrm{d}\xi\,\mathrm{d}\nu_{y,s}^2(\zeta)\\
&\qquad = \int_{\mathbb{R}^2} \varUpsilon(\xi,\zeta)\mathrm{d}\nu_{x,s}^1(\xi)\mathrm{d}\nu_{y,s}^2(\zeta)
\end{aligned} \tag{3.24}$$

where

$$\Upsilon(\xi,\zeta)=\int_{-\infty}^{\xi}\int_{\zeta}^{\infty}\Gamma(\xi',\zeta')|\xi'-\zeta'|\psi_\delta(\xi'-\zeta')\mathrm{d}\zeta'\,\mathrm{d}\xi'.$$

Therefore, we find

$$|\mathrm{I}|\leq\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\Upsilon(\xi,\zeta)\,\mathrm{d}\nu^1_{x,s}(\xi)\mathrm{d}\nu^2_{y,s}(\zeta)\,\big|\nabla_x\varrho_\tau(x-y)\big|\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.$$

The function Υ can be estimated using the substitution $\xi''=\xi'-\zeta'$

$$\begin{aligned}\Upsilon(\xi,\zeta)&=\int_{\zeta}^{\infty}\int_{|\xi''|<\delta,\,\xi''<\xi-\zeta'}\Gamma(\xi''+\zeta',\zeta')|\xi''|\psi_\delta(\xi'')\,\mathrm{d}\xi''\,\mathrm{d}\zeta'\\&\leq C\delta\int_{\zeta}^{\xi+\delta}\max_{|\xi''|<\delta,\,\xi''<\xi-\zeta'}\Gamma(\xi''+\zeta',\zeta')\,\mathrm{d}\zeta'\\&\leq C\delta\int_{\zeta}^{\xi+\delta}\big(1+|\xi|^{p-1}+|\zeta'|^{p-1}\big)\,\mathrm{d}\zeta'\\&\leq C\delta\big(1+|\xi|^p+|\zeta|^p\big)\end{aligned}$$

hence, since ν^1, ν^2 vanish at infinity,

$$|\mathrm{I}|\leq Ct\delta\int_{\mathbb{T}^N}\big|\nabla_x\varrho_\tau(x)\big|\,\mathrm{d}x\leq Ct\delta\tau^{-1}.$$

We recall that $f_1=\mathbf{1}_{u_1(x,t)>\xi}$, $f_2=\mathbf{1}_{u_2(y,t)>\zeta}$ hence

$$\partial_\xi f_1=-\nu^1=-\delta_{u_1(x,t)=\xi},\qquad\partial_\zeta f_2=-\nu^2=-\delta_{u_2(y,t)=\zeta}$$

and as both u_1, u_2 possess some regularity in the nondegeneracy zones of A due to Definition 3.2.2(ii), we obtain as in (3.11)

$$\sigma\,\nabla f_1=\sigma\,\nabla u_1\,\delta_{u_1(x,s)=\xi},\qquad\sigma\,\nabla\bar{f}_2=-\sigma\,\nabla u_2\,\delta_{u_2(y,s)=\zeta}$$

in the sense of distributions over $\mathbb{T}^N\times\mathbb{R}$. The first term in J can be rewritten in the following manner using integration by parts (and considering only relevant integrals)

$$\begin{aligned}&\int_{\mathbb{T}^N}f_1\int_{\mathbb{T}^N}\bar{f}_2\,\partial_{y_j}\big(A_{ij}(y)\partial_{y_i}\varrho_\tau(x-y)\big)\mathrm{d}y\,\mathrm{d}x\\&\qquad=\int_{(\mathbb{T}^N)^2}f_1(x,s,\xi)A_{ij}(y)\partial_{y_j}\bar{f}_2(y,s,\zeta)\partial_{x_i}\varrho_\tau(x-y)\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

and similarly for the second term. Let us define

$$\Theta_\delta(\xi)=\int_{-\infty}^{\xi}\psi_\delta(\zeta)\,\mathrm{d}\zeta.$$

Then we have $\mathrm{J} = \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3$ with

$$\begin{aligned}
\mathrm{J}_1 &= -\,\mathbb{E}\int_0^t \int_{(\mathbb{T}^N)^2} (\nabla_x u_1)^* \sigma(x)\sigma(x)(\nabla\varrho_\tau)(x-y)\Theta_\delta\big(u_1(x,s) - u_2(y,s)\big)\mathrm{d}x\mathrm{d}y\mathrm{d}s,\\
\mathrm{J}_2 &= -\,\mathbb{E}\int_0^t \int_{(\mathbb{T}^N)^2} (\nabla_y u_2)^* \sigma(y)\sigma(y)(\nabla\varrho_\tau)(x-y)\Theta_\delta\big(u_1(x,s) - u_2(y,s)\big)\mathrm{d}x\mathrm{d}y\mathrm{d}s,\\
\mathrm{J}_3 &= -\,\mathbb{E}\int_0^t \int_{(\mathbb{T}^N)^2} \big[|\sigma(x)\nabla_x u_1|^2 + |\sigma(y)\nabla_y u_2|^2\big]\varrho_\tau(x-y)\\
&\qquad\qquad \times \psi_\delta\big(u_1(x,s) - u_2(y,s)\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

Let

$$\mathrm{H} = \mathbb{E}\int_0^t \int_{(\mathbb{T}^N)^2} (\nabla_x u_1)^* \sigma(x)\sigma(y)(\nabla_y u_2)\varrho_\tau(x-y)\psi_\delta\big(u_1(x,s) - u_2(y,s)\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.$$

We intend to show that $\mathrm{J}_1 = \mathrm{H} + o(1)$, $\mathrm{J}_2 = \mathrm{H} + o(1)$, where $o(1) \to 0$ as $\tau \to 0$ uniformly in δ, and consequently

$$\begin{aligned}
\mathrm{J} = -\,\mathbb{E}\int_0^t \int_{(\mathbb{T}^N)^2} &\big|\sigma(x)\nabla_x u_1 - \sigma(y)\nabla_y u_2\big|^2 \varrho_\tau(x-y)\\
&\times \psi_\delta\big(u_1(x,s) - u_2(y,s)\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s + o(1) \le o(1).
\end{aligned} \tag{3.25}$$

We only prove the claim for J_1 since the case of J_2 is analogous. Let us define

$$g(x,y,s) = (\nabla_x u_1)^* \sigma(x)\Theta_\delta\big(u_1(x,s) - u_2(y,s)\big).$$

Here, we employ again the assumption (ii) in Definition 3.2.2. Recall, that it gives us some regularity of the solution in the nondegeneracy zones of the diffusion matrix A and hence $g \in L^2(\Omega \times \mathbb{T}^N_x \times \mathbb{T}^N_y \times [0,T])$. It holds

$$\begin{aligned}
\mathrm{J}_1 &= -\mathbb{E}\int_0^t \int_{(\mathbb{T}^N)^2} g(x,y,s)\Big(\sigma(x) - \sigma(y)\Big)(\nabla\varrho_\tau)(x-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\quad - \mathbb{E}\int_0^t \int_{(\mathbb{T}^N)^2} g(x,y,s)\sigma(y)(\nabla\varrho_\tau)(x-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s,\\
\mathrm{H} &= \mathbb{E}\int_0^t \int_{(\mathbb{T}^N)^2} g(x,y,s)\,\mathrm{div}_y\Big(\sigma(y)\varrho_\tau(x-y)\Big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&= \mathbb{E}\int_0^t \int_{(\mathbb{T}^N)^2} g(x,y,s)\,\mathrm{div}\,\big(\sigma(y)\big)\varrho_\tau(x-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\quad - \mathbb{E}\int_0^t \int_{(\mathbb{T}^N)^2} g(x,y,s)\sigma(y)(\nabla\varrho_\tau)(x-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s,
\end{aligned}$$

where divergence is applied row-wise to a matrix-valued function. Therefore, it is enough to show that the first terms in J_1 and H have the same limit value if $\tau \to 0$. For H, we obtain easily

$$\begin{aligned}
&\mathbb{E}\int_0^t \int_{(\mathbb{T}^N)^2} g(x,y,s)\,\mathrm{div}\big(\sigma(y)\big)\varrho_\tau(x-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\qquad\qquad\longrightarrow\ \mathbb{E}\int_0^t \int_{\mathbb{T}^N} g(y,y,s)\,\mathrm{div}\,\big(\sigma(y)\big)\,\mathrm{d}y\,\mathrm{d}s
\end{aligned}$$

so it remains to verify

$$-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2} g(x,y,s)\Big(\sigma(x)-\sigma(y)\Big)(\nabla\varrho_\tau)(x-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$\longrightarrow \mathbb{E}\int_0^t\int_{\mathbb{T}^N} g(y,y,s)\,\mathrm{div}\,\big(\sigma(y)\big)\,\mathrm{d}y\,\mathrm{d}s.$$

Here, we employ again the arguments of the commutation lemma of DiPerna and Lions (see [17, Lemma II.1], cf. Lemma 3.2.5). Let us denote by g^i the i^{th} element of g and by σ^i the i^{th} row of σ. Since $\tau|\nabla\varrho_\tau|(\cdot)\leq C\varrho_{2\tau}(\cdot)$ with a constant independent of τ, we obtain the following estimate

$$\begin{aligned}\mathbb{E}\int_0^t\int_{\mathbb{T}^N}&\left|\int_{\mathbb{T}^N} g^i(x,y,s)\Big(\sigma^i(x)-\sigma^i(y)\Big)(\nabla\varrho_\tau)(x-y)\,\mathrm{d}x\right|\mathrm{d}y\,\mathrm{d}s\\
&\leq C\operatorname*{ess\,sup}_{\substack{x',y'\in\mathbb{T}^N\\|x'-y'|\leq\tau}}\left|\frac{\sigma^i(x')-\sigma^i(y')}{\tau}\right|\mathbb{E}\int_0^T\int_{(\mathbb{T}^N)^2}\big|g^i(x,y,s)\big|\varrho_{2\tau}(x-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.\end{aligned}$$

Note that according to [24], [66], the square-root matrix of A is Lipschitz continuous and therefore the essential supremum can be estimated by a constant independent of τ. Next

$$\begin{aligned}\mathbb{E}\int_0^T\int_{(\mathbb{T}^N)^2}&\big|g^i(x,y,s)\big|\varrho_{2\tau}(x-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\leq\left(\mathbb{E}\int_0^T\int_{(\mathbb{T}^N)^2}\big|g^i(x,y,s)\big|^2\varrho_{2\tau}(x-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\right)^{\frac12}\\
&\qquad\times\left(\int_{(\mathbb{T}^N)^2}\varrho_{2\tau}(x-y)\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac12}\\
&\leq\left(\mathbb{E}\int_0^T\int_{\mathbb{T}^N}\big|(\nabla_x u_1)^*\sigma(x)\big|^2\int_{\mathbb{T}^N}\varrho_{2\tau}(x-y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac12}\\
&\leq\big\|(\nabla_x u_1)^*\sigma(x)\big\|_{L^2(\Omega\times\mathbb{T}^N\times[0,T])}.\end{aligned}$$

So we get an estimate which is independent of τ and δ. It is sufficient to consider the case when g^i and σ^i are smooth. The general case follows by density argument from the above bound. It holds

$$\begin{aligned}-\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}&g^i(x,y,s)\Big(\sigma^i(x)-\sigma^i(y)\Big)(\nabla\varrho_\tau)(x-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&=-\frac{1}{\tau^{N+1}}\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_0^1 g^i(x,y,s)\,\mathrm{D}\sigma^i\big(y+r(x-y)\big)(x-y)\\
&\qquad\qquad\cdot(\nabla\varrho)\Big(\frac{x-y}{\tau}\Big)\,\mathrm{d}r\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&=-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\int_0^1 g^i(y+\tau z,y,s)\,\mathrm{D}\sigma^i(y+r\tau z)z\cdot(\nabla\varrho)(z)\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s\\
\longrightarrow\ &-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}g^i(y,y,s)\,\mathrm{D}\sigma^i(y)z\cdot(\nabla\varrho)(z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s.\end{aligned}$$

Moreover, by (3.14),

$$-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2} g^i(y,y,s)\,\mathrm{D}\sigma^i(y)z\cdot(\nabla\varrho)(z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s$$
$$=\mathbb{E}\int_0^t\int_{\mathbb{T}^N} g^i(y,y,s)\,\mathrm{div}\big(\sigma^i(y)\big)\,\mathrm{d}y\,\mathrm{d}s$$

and accordingly (3.25) follows.

The last term K is, due to (3.3), bounded as follows

$$\begin{aligned}
\mathrm{K}&\leq\frac{L}{2}\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\varrho_\tau(x-y)|x-y|^2\int_{\mathbb{R}^2}\psi_\delta(\xi-\zeta)\,\mathrm{d}\nu^1_{x,s}(\xi)\,\mathrm{d}\nu^2_{y,s}(\zeta)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\quad+\frac{L}{2}\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\varrho_\tau(x-y)\int_{\mathbb{R}^2}\psi_\delta(\xi-\zeta)|\xi-\zeta|h(|\xi-\zeta|)\mathrm{d}\nu^1_{x,s}(\xi)\mathrm{d}\nu^2_{y,s}(\zeta)\mathrm{d}x\mathrm{d}y\mathrm{d}s\\
&\leq\frac{Lt}{2\delta}\int_{(\mathbb{T}^N)^2}|x-y|^2\varrho_\tau(x-y)\,\mathrm{d}x\,\mathrm{d}y+\frac{LtC_\psi h(\delta)}{2}\int_{(\mathbb{T}^N)^2}\varrho_\tau(x-y)\,\mathrm{d}x\,\mathrm{d}y\\
&\leq\frac{Lt}{2}\delta^{-1}\tau^2+\frac{LtC_\psi h(\delta)}{2},
\end{aligned}$$

where $C_\psi=\sup_{\xi\in\mathbb{R}}|\xi\psi(\xi)|$. Finally, we set $\delta=\tau^{4/3}$, let $\tau\to0$ and deduce

$$\mathbb{E}\int_{\mathbb{T}^N}\int_{\mathbb{R}}f_1^\pm(t)\bar{f}_2^\pm(t)\,\mathrm{d}\xi\,\mathrm{d}x\leq\mathbb{E}\int_{\mathbb{T}^N}\int_{\mathbb{R}}f_{1,0}\bar{f}_{2,0}\,\mathrm{d}\xi\,\mathrm{d}x.$$

Let us now consider $f_1=f_2=f$. Since $f_0=\mathbf{1}_{u_0>\xi}$ we have the identity $f_0\bar{f}_0=0$ and therefore $f^\pm(1-f^\pm)=0$ a.e. (ω,x,ξ) and for all t. The fact that $f^\pm$ is a kinetic function and Fubini's theorem then imply that, for any $t\in[0,T]$, there exists a set $\Sigma_t\subset\Omega\times\mathbb{T}^N$ of full measure such that, for $(\omega,x)\in\Sigma_t$, $f^\pm(\omega,x,t,\xi)\in\{0,1\}$ for a.e. $\xi\in\mathbb{R}$. Therefore, there exist $u^\pm:\Omega\times\mathbb{T}^N\times[0,T]\to\mathbb{R}$ such that $f^\pm=\mathbf{1}_{u^\pm>\xi}$ for a.e (ω,x,ξ) and all t. In particular, $u^\pm=\int_{\mathbb{R}}(f^\pm-\mathbf{1}_{0>\xi})\mathrm{d}\xi$ for a.e. (ω,x) and all t. It follows now from Proposition 3.3.1 and the identity

$$|\alpha-\beta|=\int_{\mathbb{R}}|\mathbf{1}_{\alpha>\xi}-\mathbf{1}_{\beta>\xi}|\,\mathrm{d}\xi,\qquad\alpha,\,\beta\in\mathbb{R},$$

that $u^+=u^-=u$ for a.e. $t\in[0,T]$. Since

$$\int_{\mathbb{R}}\mathbf{1}_{u_1^\pm>\xi}\overline{\mathbf{1}_{u_2^\pm>\xi}}\,\mathrm{d}\xi=(u_1^\pm-u_2^\pm)^+$$

we obtain the comparison property

$$\mathbb{E}\big\|\big(u_1^\pm(t)-u_2^\pm(t)\big)^+\big\|_{L^1(\mathbb{T}^N)}\leq\mathbb{E}\big\|(u_{1,0}-u_{2,0})^+\big\|_{L^1(\mathbb{T}^N)}.$$

□

As a consequence of Theorem 3.3.3, namely from the comparison property (3.23), the uniqueness part of Theorem 3.2.10 follows. Furthermore, we obtain the continuity of trajectories in $L^p(\mathbb{T}^N)$.

Corollary 3.3.4 (Continuity in time)**.** *Let u be a kinetic solution to* (3.1)*. Then there exists a representative of u which has almost surely continuous trajectories in $L^p(\mathbb{T}^N)$, for all $p\in[1,\infty)$.*

Proof. Remark, that due to the construction of $f^{\pm}$ it holds, for all $p\in[1,\infty)$,

$$\mathbb{E}\sup_{0\le t\le T}\int_{\mathbb{T}^N}|u^{\pm}(x,t)|^p\,\mathrm{d}x=\mathbb{E}\sup_{0\le t\le T}\int_{\mathbb{T}^N}\int_{\mathbb{R}}|\xi|^p\,\mathrm{d}\nu^{\pm}_{x,t}(\xi)\,\mathrm{d}x\le C. \tag{3.26}$$

Now, we are able to prove that the modification u^+ is right-continuous in the sense of $L^p(\mathbb{T}^N)$. According to the Proposition 3.3.1 applied to the solution f^+, we obtain

$$\big\langle f^+(t+\varepsilon),\psi\big\rangle\longrightarrow\big\langle f^+(t),\psi\big\rangle,\qquad\varepsilon\downarrow 0,\qquad\forall\psi\in L^1(\mathbb{T}^N\times\mathbb{R}).$$

Setting $\psi(x,\xi)=\psi_1(x)\partial_\xi\psi_2(\xi)$ for some functions $\psi_1\in L^1(\mathbb{T}^N)$ and $\partial_\xi\psi_2\in C^\infty_c(\mathbb{R})$, it reads

$$\int_{\mathbb{T}^N}\psi_1(x)\psi_2\big(u^+(x,t+\varepsilon)\big)\,\mathrm{d}x\longrightarrow\int_{\mathbb{T}^N}\psi_1(x)\psi_2\big(u^+(x,t)\big)\,\mathrm{d}x. \tag{3.27}$$

In order to obtain that $u^+(t+\varepsilon)\xrightarrow{w}u^+(t)$ in $L^p(\mathbb{T}^N)$, $p\in[1,\infty)$, we set $\psi_2^\delta(\xi)=\xi\chi_\delta(\xi)$ where (χ_δ) is a truncation on $\mathbb{R}$, i.e. we define $\chi_\delta(\xi)=\chi(\delta\xi)$, where χ is a smooth function with bounded support satisfying $0\le\chi\le 1$ and

$$\chi(\xi)=\begin{cases}1, & \text{if }\ |\xi|\le\frac12,\\ 0, & \text{if }\ |\xi|\ge 1,\end{cases}$$

and deduce

$$\begin{aligned}&\bigg|\int_{\mathbb{T}^N}\psi_1(x)\,u^+(x,t+\varepsilon)\,\mathrm{d}x-\int_{\mathbb{T}^N}\psi_1(x)\,u^+(x,t)\,\mathrm{d}x\bigg|\\
&\quad\le\int_{\mathbb{T}^N}\big|\psi_1(x)\,u^+(x,t+\varepsilon)\big|\,\mathbf{1}_{|u^+(x,t+\varepsilon)|>1/2\delta}\,\mathrm{d}x\\
&\qquad+\bigg|\int_{\mathbb{T}^N}\psi_1(x)\psi_2^\delta\big(u^+(x,t+\varepsilon)\big)-\psi_1(x)\psi_2^\delta\big(u^+(x,t)\big)\,\mathrm{d}x\bigg|\\
&\qquad+\int_{\mathbb{T}^N}\big|\psi_1(x)\,u^+(x,t)\big|\,\mathbf{1}_{|u^+(x,t)|>1/2\delta}\,\mathrm{d}x\longrightarrow 0,\qquad\varepsilon\downarrow 0,\end{aligned}$$

since the first and the third term on the right hand side tend to zero as $\delta\to 0$ uniformly in ε due to the uniform estimate (3.26) and the second one vanishes as $\varepsilon\to 0$ for any δ by (3.27).

The strong convergence in $L^2(\mathbb{T}^N)$ then follows easily as soon as we verify the convergence of the $L^2(\mathbb{T}^N)$-norms. This can be done by a similar approximation procedure, using $\psi_1(x)=1$ and $\psi_2^\delta(\xi)=\xi^2\chi_\delta(\xi)$. For the strong convergence in $L^p(\mathbb{T}^N)$ for general $p\in[1,\infty)$ we employ the Hölder inequality and the uniform bound (3.26).

A similar approach then shows that the modification u^- is left-continuous in the sense of $L^p(\mathbb{T}^N)$. The rest of the proof, showing that $u^-(t)=u^+(t)$ for all $t\in[0,T]$ can be carried out similarly to [16, Corollary 12]. □

3.4 Existence - smooth initial data

In this section we prove the existence part of Theorem 3.2.10 under an additional assumption upon the initial condition: $u_0\in L^p(\Omega;C^\infty(\mathbb{T}^N))$, for all $p\in[1,\infty)$. We employ the vanishing viscosity method, i.e. we approximate the equation (3.1) by certain nondegenerate problems, while using also some appropriately chosen approximations Φ^ε, B^ε of Φ and B, respectively. These equations have smooth solutions and consequent passage to the limit gives the existence of a kinetic solution to the original equation. Nevertheless, the limit argument is quite technical and has to be done in several steps. It is based on the compactness method: the uniform energy estimates yield tightness of a sequence of approximate solutions and thus, on another probability space, this sequence converges almost surely due to the Skorokhod representation theorem. The limit is then shown to be a martingale kinetic solution to (3.1). Combining this fact and the pathwise uniqueness with the Gyöngy-Krylov characterization of convergence in probability, we finally obtain the desired kinetic solution.

3.4.1 Nondegenerate case

Consider a truncation (χ_ε) on $\mathbb{R}$ and approximations to the identity (φ_ε), (ψ_ε) on $\mathbb{T}^N\times\mathbb{R}$ and $\mathbb{R}$, respectively. To be more precise concerning the case of $\mathbb{T}^N\times\mathbb{R}$, we make use of the same notation as at the beginning of the proof of Theorem 3.3.3 and define

$$\varphi_\varepsilon(x,\xi)=\frac{1}{\varepsilon^{N+1}}\varrho\Big(\frac{x}{\varepsilon}\Big)\psi\Big(\frac{\xi}{\varepsilon}\Big).$$

The regularizations of $\varPhi$, B are then defined in the following way

$$B_i^\varepsilon(\xi)=\big((B_i*\psi_\varepsilon)\chi_\varepsilon\big)(\xi),\qquad i=1,\dots,N,$$

$$g_k^\varepsilon(x,\xi)=\begin{cases}\big((g_k*\varphi_\varepsilon)\chi_\varepsilon\big)(x,\xi), & \text{if } \ k\leq\lfloor 1/\varepsilon\rfloor,\\ 0, & \text{if } \ k>\lfloor 1/\varepsilon\rfloor,\end{cases}$$

where $x\in\mathbb{T}^N,\xi\in\mathbb{R}$. Consequently, we set $B^\varepsilon=(B_1^\varepsilon,\dots,B_N^\varepsilon)$ and define the operator $\varPhi^\varepsilon$ by $\varPhi^\varepsilon(z)e_k=g_k^\varepsilon(\cdot,z(\cdot))$, $z\in L^2(\mathbb{T}^N)$. Clearly, the approximations B^ε, g_k^ε are of class C^∞ with a compact support therefore Lipschitz continuous. Moreover, the functions g_k^ε satisfy (3.2), (3.3) uniformly in ε and the following Lipschitz condition holds true

$$\forall x\in\mathbb{T}^N\quad\forall\xi,\,\zeta\in\mathbb{R}\quad\sum_{k\geq1}|g_k^\varepsilon(x,\xi)-g_k^\varepsilon(x,\zeta)|^2\leq L_\varepsilon|\xi-\zeta|^2. \tag{3.28}$$

From (3.2) we conclude that $\varPhi^\varepsilon(z)$ is Hilbert-Schmidt for all $z\in L^2(\mathbb{T}^N)$. Also the polynomial growth of B remains valid for B^ε and holds uniformly in ε. Suitable approximation of the diffusion matrix A is obtained as its perturbation by $\varepsilon\mathrm{I}$, where I denotes the identity matrix. We denote $A^\varepsilon=A+\varepsilon\mathrm{I}$.

Consider an approximation of problem (3.1) by a nondegenerate equation

$$\begin{aligned}\mathrm{d}u^\varepsilon+\operatorname{div}\big(B^\varepsilon(u^\varepsilon)\big)\mathrm{d}t&=\operatorname{div}\big(A^\varepsilon(x)\nabla u^\varepsilon\big)\mathrm{d}t+\varPhi^\varepsilon(u^\varepsilon)\,\mathrm{d}W,\\ u^\varepsilon(0)&=u_0.\end{aligned}\tag{3.29}$$

Theorem 3.4.1. *Assume that $u_0\in L^p(\Omega;C^\infty(\mathbb{T}^N))$ for all $p\in(2,\infty)$. For any $\varepsilon>0$, there exists a $C^\infty(\mathbb{T}^N)$-valued process which is the unique strong solution to (3.29). Moreover, it belongs to*

$$L^p(\Omega;C([0,T];W^{l,q}(\mathbb{T}^N)))\qquad\textit{for every}\quad p\in(2,\infty),\,q\in[2,\infty),\,l\in\mathbb{N}.$$

Proof. For any fixed $\varepsilon>0$, the assumptions of [32, Theorem 2.1, Corollary 2.2] are satisfied and therefore the claim follows. □

Let m^ε be the parabolic dissipative measure corresponding to the diffusion matrix $A+\varepsilon I$. To be more precise, set

$$\begin{aligned}\mathrm{d}n_1^\varepsilon(x,t,\xi)&=\big|\sigma(x)\nabla u^\varepsilon\big|^2\,\mathrm{d}\delta_{u^\varepsilon(x,t)}(\xi)\,\mathrm{d}x\,\mathrm{d}t,\\ \mathrm{d}n_2^\varepsilon(x,t,\xi)&=\varepsilon|\nabla u^\varepsilon|^2\,\mathrm{d}\delta_{u^\varepsilon(x,t)}(\xi)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

and define $m^\varepsilon=n_1^\varepsilon+n_2^\varepsilon$. Then, using the same approach as in Section 3.2, one can verify that the pair $(f^\varepsilon=\mathbf{1}_{u^\varepsilon>\xi},m^\varepsilon)$ satisfies the kinetic formulation of (3.29): let $\varphi\in C_c^\infty(\mathbb{T}^N\times\mathbb{R})$, $t\in[0,T]$, then it holds true $\mathbb{P}$-a.s.

$$\begin{aligned}&\big\langle f^\varepsilon(t),\varphi\big\rangle-\big\langle f_0,\varphi\big\rangle-\int_0^t\big\langle f^\varepsilon(s),b^\varepsilon(\xi)\cdot\nabla\varphi\big\rangle\,\mathrm{d}s\\ &\qquad-\int_0^t\Big\langle f^\varepsilon(s),\operatorname{div}\big(A(x)\nabla\varphi\big)\Big\rangle\,\mathrm{d}s-\varepsilon\int_0^t\big\langle f^\varepsilon(s),\Delta\varphi\big\rangle\,\mathrm{d}s\\ &=\int_0^t\big\langle\delta_{u^\varepsilon=\xi}\,\varPhi^\varepsilon(u^\varepsilon)\,\mathrm{d}W,\varphi\big\rangle+\frac12\int_0^t\big\langle\delta_{u^\varepsilon=\xi}\,G^2,\partial_\xi\varphi\big\rangle\,\mathrm{d}s-\big\langle m^\varepsilon,\partial_\xi\varphi\big\rangle([0,t)).\end{aligned}\tag{3.30}$$

Note, that by taking limit in ε we lose this precise structure of n_2.

3.4.2 Energy estimates

In this subsection we shall establish the so-called energy estimate that makes it possible to find uniform bounds for approximate solutions and that will later on yield a solution by invoking a compactness argument.

Lemma 3.4.2. *For all $\varepsilon\in(0,1)$, for all $t\in[0,T]$ and for all $p\in[2,\infty)$, the solution u^ε satisfies the inequality*

$$\mathbb{E}\|u^\varepsilon(t)\|^p_{L^p(\mathbb{T}^N)}\leq C\big(1+\mathbb{E}\|u_0\|^p_{L^p(\mathbb{T}^N)}\big).\tag{3.31}$$

Proof. According to Theorem 3.4.1, the process u^ε is an $L^p(\mathbb{T}^N)$-valued continuous semimartingale so we can apply the infinite-dimensional Itô formula [15, Theorem 4.17] for the function $f(v)=\|v\|^p_{L^p(\mathbb{T}^N)}$. If q is the conjugate exponent to p then $f'(v)=p|v|^{p-2}v\in L^q(\mathbb{T}^N)$ and

$$f''(v)=p(p-1)|v|^{p-2}\,\mathrm{Id}\quad\in\mathscr{L}\big(L^p(\mathbb{T}^N),L^q(\mathbb{T}^N)\big).$$

Therefore

$$\begin{aligned}\|u^\varepsilon(t)\|^p_{L^p(\mathbb{T}^N)}=&\|u_0\|^p_{L^p(\mathbb{T}^N)}-p\int_0^t\int_{\mathbb{T}^N}|u^\varepsilon|^{p-2}u^\varepsilon\operatorname{div}\big(B^\varepsilon(u^\varepsilon)\big)\,\mathrm{d}x\,\mathrm{d}s\\ &+p\int_0^t\int_{\mathbb{T}^N}|u^\varepsilon|^{p-2}u^\varepsilon\operatorname{div}\big(A(x)\nabla u^\varepsilon\big)\,\mathrm{d}x\,\mathrm{d}s\\ &+\varepsilon p\int_0^t\int_{\mathbb{T}^N}|u^\varepsilon|^{p-2}u^\varepsilon\Delta u^\varepsilon\,\mathrm{d}x\,\mathrm{d}s\\ &+p\sum_{k\geq1}\int_0^t\int_{\mathbb{T}^N}|u^\varepsilon|^{p-2}u^\varepsilon g_k^\varepsilon(x,u^\varepsilon)\,\mathrm{d}x\,\mathrm{d}\beta_k(s)\\ &+\frac12p(p-1)\int_0^t\int_{\mathbb{T}^N}|u^\varepsilon|^{p-2}G_\varepsilon^2(x,u^\varepsilon)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}\tag{3.32}$$

If we define $H^\varepsilon(\xi) = \int_0^\xi |\zeta|^{p-2} B^\varepsilon(\zeta)\,\mathrm{d}\zeta$ then the second term on the right hand side vanishes due to the boundary conditions

$$-p\int_0^t\int_{\mathbb{T}^N}|u^\varepsilon|^{p-2}u^\varepsilon\operatorname{div}\big(B(u^\varepsilon)\big)\mathrm{d}x\,\mathrm{d}s = p\int_0^t\int_{\mathbb{T}^N}\operatorname{div}\big(H^\varepsilon(u^\varepsilon)\big)\,\mathrm{d}x\,\mathrm{d}s = 0.$$

The third term is nonpositive as the matrix A is positive-semidefinite

$$\begin{aligned} p\int_0^t\int_{\mathbb{T}^N}&|u^\varepsilon|^{p-2}u^\varepsilon\operatorname{div}\big(A(u^\varepsilon)\nabla u^\varepsilon\big)\mathrm{d}x\,\mathrm{d}s\\ &= -p\int_0^t\int_{\mathbb{T}^N}|u^\varepsilon|^{p-2}\big(\nabla u^\varepsilon\big)^*A(x)\big(\nabla u^\varepsilon\big)\,\mathrm{d}x\,\mathrm{d}s\le 0\end{aligned}$$

and the same holds for the fourth term as well since A is only replaced by εI. The last term is estimated as follows

$$\begin{aligned}\frac{1}{2}p(p-1)\int_0^t\int_{\mathbb{T}^N}|u^\varepsilon|^{p-2}G_\varepsilon^2(x,u^\varepsilon)\,\mathrm{d}x\,\mathrm{d}s&\le C\int_0^t\int_{\mathbb{T}^N}|u^\varepsilon|^{p-2}\big(1+|u^\varepsilon|^2\big)\,\mathrm{d}x\,\mathrm{d}s\\ &\le C\Big(1+\int_0^t\|u^\varepsilon(s)\|^p_{L^p(\mathbb{T}^N)}\mathrm{d}s\Big).\end{aligned}$$

Finally, expectation and application of the Gronwall lemma yield (3.31). □

Corollary 3.4.3. *The set $\{u^\varepsilon;\,\varepsilon\in(0,1)\}$ is bounded in $L^p(\Omega;C([0,T];L^p(\mathbb{T}^N)))$, for all $p\in[2,\infty)$.*

Proof. Continuity of trajectories follows from Theorem 3.4.1. To verify the claim, an uniform estimate of $\mathbb{E}\sup_{0\le t\le T}\|u^\varepsilon(t)\|^p_{L^p(\mathbb{T}^N)}$ is needed. We repeat the approach from the preceding lemma, only for the stochastically forced term we apply the Burkholder-Davis-Gundy inequality. We have

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\|u^\varepsilon(t)\|^p_{L^p(\mathbb{T}^N)}&\le\mathbb{E}\|u_0\|^p_{L^p(\mathbb{T}^N)}+C\Big(1+\int_0^T\mathbb{E}\|u^\varepsilon(s)\|^p_{L^p(\mathbb{T}^N)}\mathrm{d}s\Big)\\ &\quad+p\,\mathbb{E}\sup_{0\le t\le T}\Big|\sum_{k\ge1}\int_0^t\int_{\mathbb{T}^N}|u^\varepsilon|^{p-2}u^\varepsilon g_k^\varepsilon(x,u^\varepsilon)\,\mathrm{d}x\,\mathrm{d}\beta_k(s)\Big|\end{aligned}$$

and using the Burkholder-Davis-Gundy and the Schwartz inequality, the assumption (3.2) and the weighted Young inequality in the last step yield

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq T}\bigg|\sum_{k\geq 1}\int_0^t\int_{\mathbb{T}^N}|u^\varepsilon|^{p-2}u^\varepsilon g_k^\varepsilon(x,u^\varepsilon)\,\mathrm{d}x\,\mathrm{d}\beta_k(s)\bigg|\\
&\quad\leq C\,\mathbb{E}\bigg(\int_0^T\sum_{k\geq 1}\Big(\int_{\mathbb{T}^N}|u^\varepsilon|^{p-1}|g_k^\varepsilon(x,u^\varepsilon)|\,\mathrm{d}x\Big)^2\mathrm{d}s\bigg)^{\frac12}\\
&\quad\leq C\,\mathbb{E}\bigg(\int_0^T\big\|\,|u^\varepsilon|^{\frac p2}\big\|^2_{L^2(\mathbb{T}^N)}\sum_{k\geq 1}\big\|\,|u^\varepsilon|^{\frac{p-2}{2}}|g_k^\varepsilon(\cdot,u^\varepsilon(\cdot))|\big\|^2_{L^2(\mathbb{T}^N)}\,\mathrm{d}s\bigg)^{\frac12}\\
&\quad\leq C\,\mathbb{E}\bigg(\int_0^T\|u^\varepsilon\|^p_{L^p(\mathbb{T}^N)}\Big(1+\|u^\varepsilon\|^p_{L^p(\mathbb{T}^N)}\Big)\,\mathrm{d}s\bigg)^{\frac12}\\
&\quad\leq C\,\mathbb{E}\bigg(\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^p_{L^p(\mathbb{T}^N)}\bigg)^{\frac12}\bigg(1+\int_0^T\|u^\varepsilon(s)\|^p_{L^p(\mathbb{T}^N)}\,\mathrm{d}s\bigg)^{\frac12}\\
&\quad\leq \frac12\,\mathbb{E}\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^p_{L^p(\mathbb{T}^N)}+C\bigg(1+\int_0^T\mathbb{E}\|u^\varepsilon(s)\|^p_{L^p(\mathbb{T}^N)}\,\mathrm{d}s\bigg).
\end{aligned}$$

Therefore

$$\mathbb{E}\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^p_{L^p(\mathbb{T}^N)}\leq C\bigg(1+\mathbb{E}\|u_0\|^p_{L^p(\mathbb{T}^N)}+\int_0^T\mathbb{E}\|u^\varepsilon(s)\|^p_{L^p(\mathbb{T}^N)}\,\mathrm{d}s\bigg)$$

and the corollary follows from (3.31). □

3.4.3 Compactness argument

To show that there exists $u:\Omega\times\mathbb{T}^N\times[0,T]\to\mathbb{R}$, a kinetic solution to (3.1), one needs to verify the strong convergence of the approximate solutions u^ε. This can be done by combining tightness of their laws with the pathwise uniqueness, which was proved above.

First, we need to prove a better spatial regularity of the approximate solutions. Towards this end, we introduce two seminorms describing the $W^{\lambda,1}$-regularity of a function $u\in L^1(\mathbb{T}^N)$. Let $\lambda\in(0,1)$ and define

$$\begin{aligned}
p^\lambda(u)&=\int_{\mathbb{T}^N}\int_{\mathbb{T}^N}\frac{|u(x)-u(y)|}{|x-y|^{N+\lambda}}\,\mathrm{d}x\,\mathrm{d}y,\\
p^\lambda_\varrho(u)&=\sup_{0<\tau<2D_N}\frac{1}{\tau^\lambda}\int_{\mathbb{T}^N}\int_{\mathbb{T}^N}|u(x)-u(y)|\varrho_\tau(x-y)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

where (ϱ_τ) is the approximation to the identity on $\mathbb{T}^N$ (as introduced in the proof of Lemma 3.2.5) that is radial, i.e. $\varrho_\tau(x)=1/\tau^N\varrho(|x|/\tau)$; and by D_N we denote the diameter of $[0,1]^N$. The fractional Sobolev space $W^{\lambda,1}(\mathbb{T}^N)$ is defined as a subspace of $L^1(\mathbb{T}^N)$ with finite norm

$$\|u\|_{W^{\lambda,1}(\mathbb{T}^N)}=\|u\|_{L^1(\mathbb{T}^N)}+p^\lambda(u).$$

According to [16], the following relations holds true between these seminorms. Let $s\in(0,\lambda)$, there exists a constant $C=C_{\lambda,\varrho,N}$ such that for all $u\in L^1(\mathbb{T}^N)$

$$p^\lambda_\varrho(u)\leq Cp^\lambda(u),\qquad p^s(u)\leq\frac{C}{\lambda-s}\,p^\lambda_\varrho(u). \tag{3.33}$$

Theorem 3.4.4 ($W^{\varsigma,1}$-regularity). *Set $\varsigma=\min\{\frac{\alpha}{\alpha+1},\frac12\}$, where α was introduced in (3.4). Then for all $s\in(0,\varsigma)$ there exists a constant $C_{T,s}>0$ such that for all $t\in[0,T]$ and all $\varepsilon\in(0,1)$*

$$\mathbb{E}\,p^s\big(u^\varepsilon(t)\big)\leq C_{T,s}\big(1+\mathbb{E}\,p^\varsigma(u_0)\big). \tag{3.34}$$

In particular, there exists a constant $C_{T,s,u_0}>0$ such that for all $t\in[0,T]$

$$\mathbb{E}\|u^\varepsilon(t)\|_{W^{s,1}(\mathbb{T}^N)}\leq C_{T,s,u_0}\big(1+\mathbb{E}\|u_0\|_{W^{\varsigma,1}(\mathbb{T}^N)}\big). \tag{3.35}$$

Proof. Proof of this statement is based on Proposition 3.3.2. We have

$$\begin{aligned}
&\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\varrho_\tau(x-y)f^\varepsilon(x,t,\xi)\bar{f}^\varepsilon(y,t,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y\\
&\leq\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho_\tau(x-y)\psi_\delta(\xi-\zeta)f^\varepsilon(x,t,\xi)\bar{f}^\varepsilon(y,t,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t+\delta\\
&\leq\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\varrho_\tau(x-y)\psi_\delta(\xi-\zeta)f_0(x,\xi)\bar{f}_0(y,\zeta)\mathrm{d}\xi\mathrm{d}\zeta\mathrm{d}x\mathrm{d}y+\delta+\mathrm{I}^\varepsilon+\mathrm{J}^\varepsilon+\mathrm{K}^\varepsilon\\
&\leq\mathbb{E}\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}}\varrho_\tau(x-y)f_0(x,\xi)\bar{f}_0(y,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y+2\delta+\mathrm{I}^\varepsilon+\mathrm{J}^\varepsilon+\mathrm{K}^\varepsilon,
\end{aligned}$$

where $\mathrm{I}^\varepsilon,\mathrm{J}^\varepsilon,\mathrm{K}^\varepsilon$ are defined correspondingly to $\mathrm{I},\mathrm{J},\mathrm{K}$ in Proposition 3.3.2 but using the approximated coefficients $B^\varepsilon,A^\varepsilon,\varPhi^\varepsilon$ instead. From the same estimates as the ones used in the proof of Theorem 3.3.3, we conclude

$$\begin{aligned}
&\mathbb{E}\int_{(\mathbb{T}^N)^2}\varrho_\tau(x-y)\big(u^\varepsilon(x,t)-u^\varepsilon(y,t)\big)^+\,\mathrm{d}x\,\mathrm{d}y\\
&\leq\mathbb{E}\int_{(\mathbb{T}^N)^2}\varrho_\tau(x-y)\big(u_0(x)-u_0(y)\big)^+\,\mathrm{d}x\mathrm{d}y+2\delta+Ct\big(\delta^{-1}\tau+\delta^{-1}\tau^2+\delta^\alpha\big)+\mathrm{J}^\varepsilon.
\end{aligned}$$

In order to control the term J^ε, recall that (keeping the notation from Theorem 3.3.3)

$$\begin{aligned}
\mathrm{J}^\varepsilon=&-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}(\nabla_x u^\varepsilon)^*\sigma(x)\sigma(x)(\nabla\varrho_\tau)(x-y)\Theta_\delta\big(u^\varepsilon(x)-u^\varepsilon(y)\big)\mathrm{d}x\mathrm{d}y\mathrm{d}r,\\
&-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}(\nabla_y u^\varepsilon)^*\sigma(y)\sigma(y)(\nabla\varrho_\tau)(x-y)\Theta_\delta\big(u^\varepsilon(x)-u^\varepsilon(y)\big)\mathrm{d}x\mathrm{d}y\mathrm{d}r,\\
&-\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\big[|\sigma(x)\nabla_x u^\varepsilon|^2+|\sigma(y)\nabla_y u^\varepsilon|^2\big]\varrho_\tau(x-y)\\
&\qquad\qquad\times\psi_\delta\big(u^\varepsilon(x)-u^\varepsilon(y)\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\\
&-\varepsilon\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\big|\nabla_x u^\varepsilon-\nabla_y u^\varepsilon\big|^2\varrho_\tau(x-y)\psi_\delta\big(u^\varepsilon(x)-u^\varepsilon(y)\big)\,\mathrm{d}x\mathrm{d}y\mathrm{d}r\\
=&\;\mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3+\mathrm{J}_4.
\end{aligned}$$

The first three terms on the above right hand side correspond to the diffusion term $\operatorname{div}(A(x)\nabla u^\varepsilon)$. Since all u^ε are smooth and hence the chain rule formula is not an issue here, J_4 is obtained after integration by parts from similar terms corresponding to $\varepsilon\Delta u^\varepsilon$. Next, we have

$$\begin{aligned}\mathrm{J}_1 =& -\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}(\nabla_x u^\varepsilon)^*\sigma(x)\,\mathrm{div}_y\Big(\sigma(y)\Theta_\delta\big(u^\varepsilon(x)-u^\varepsilon(y)\big)\Big)\varrho_\tau(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}r\\ &+\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}(\nabla_x u^\varepsilon)^*\sigma(x)\Big(\sigma(y)-\sigma(x)\Big)(\nabla\varrho_\tau)(x-y)\Theta_\delta\big(u^\varepsilon(x)-u^\varepsilon(y)\big)\mathrm{d}x\mathrm{d}y\mathrm{d}r\end{aligned}$$

and

$$\begin{aligned}\mathrm{J}_2 =& \,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}(\nabla_y u^\varepsilon)^*\sigma(y)\,\mathrm{div}_x\Big(\sigma(x)\Theta_\delta\big(u^\varepsilon(x)-u^\varepsilon(y)\big)\Big)\varrho_\tau(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}r\\ &+\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}(\nabla_y u^\varepsilon)^*\sigma(y)\Big(\sigma(x)-\sigma(y)\Big)(\nabla\varrho_\tau)(x-y)\Theta_\delta\big(u^\varepsilon(x)-u^\varepsilon(y)\big)\mathrm{d}x\mathrm{d}y\mathrm{d}r\end{aligned}$$

hence $\mathrm{J}_1=\mathrm{H}+\mathrm{R}_1$ and $\mathrm{J}_2=\mathrm{H}+\mathrm{R}_2$ where

$$\begin{aligned}\mathrm{H} &= \mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}(\nabla_x u^\varepsilon)^*\sigma(x)\sigma(y)(\nabla_y u^\varepsilon)\varrho_\tau(x-y)\psi_\delta\big(u^\varepsilon(x)-u^\varepsilon(y)\big)\mathrm{d}x\mathrm{d}y\mathrm{d}r\\ \mathrm{R}_1 &= \mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}(\nabla_x u^\varepsilon)^*\sigma(x)\Theta_\delta\big(u^\varepsilon(x)-u^\varepsilon(y)\big)\\ &\qquad\times\Big(\big(\sigma(y)-\sigma(x)\big)(\nabla\varrho_\tau)(x-y)-\mathrm{div}\,\big(\sigma(y)\big)\varrho_\tau(x-y)\Big)\mathrm{d}x\mathrm{d}y\mathrm{d}r\\ \mathrm{R}_2 &= \mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}(\nabla_y u^\varepsilon)^*\sigma(y)\Theta_\delta\big(u^\varepsilon(x)-u^\varepsilon(y)\big)\\ &\qquad\times\Big(\big(\sigma(x)-\sigma(y)\big)(\nabla\varrho_\tau)(x-y)+\mathrm{div}\,\big(\sigma(x)\big)\varrho_\tau(x-y)\Big)\mathrm{d}x\mathrm{d}y\mathrm{d}r.\end{aligned}$$

As a consequence, we see that $\mathrm{J}^\varepsilon=\mathrm{J}_4+\mathrm{J}_5+\mathrm{R}_1+\mathrm{R}_2$ where

$$\begin{aligned}\mathrm{J}_5 = -\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}&\big|\sigma(x)\nabla_x u^\varepsilon-\sigma(y)\nabla_y u^\varepsilon\big|^2\varrho_t(x-y)\\ &\times\psi_\delta\big(u^\varepsilon(x)-u^\varepsilon(y)\big)\,\mathrm{d}x\mathrm{d}y\mathrm{d}r\end{aligned}$$

and therefore $\mathrm{J}^\varepsilon\leq\mathrm{R}_1+\mathrm{R}_2$. Let us introduce an auxiliary function

$$T_\delta(\xi)=\int_0^\xi\Theta_\delta(\zeta)\,\mathrm{d}\zeta.$$

With this in hand we obtain

$$\begin{aligned}\mathrm{R}_1 &= \mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\sigma(x)\nabla_x T_\delta\big(u^\varepsilon(x)-u^\varepsilon(y)\big)\\ &\qquad\times\Big(\big(\sigma(y)-\sigma(x)\big)(\nabla\varrho_\tau)(x-y)-\mathrm{div}\,\big(\sigma(y)\big)\varrho_\tau(x-y)\Big)\mathrm{d}x\mathrm{d}y\mathrm{d}r\\ &= -\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}T_\delta\big(u^\varepsilon(x)-u^\varepsilon(y)\big)\Big[\,\mathrm{div}\,\big(\sigma(x)\big)\big(\sigma(y)-\sigma(x)\big)(\nabla\varrho_\tau)(x-y)\\ &\quad-\sigma(x)\,\mathrm{div}\,\big(\sigma(x)\big)(\nabla\varrho_\tau)(x-y)+\sigma(x)\big(\sigma(y)-\sigma(x)\big)(\nabla^2\varrho_\tau)(x-y)\\ &\quad-\mathrm{div}\,\big(\sigma(x)\big)\,\mathrm{div}\,\big(\sigma(y)\big)\varrho_\tau(x-y)-\sigma(x)\,\mathrm{div}\,\big(\sigma(y)\big)(\nabla\varrho_\tau)(x-y)\Big]\mathrm{d}x\mathrm{d}y\mathrm{d}r\end{aligned}$$

and similarly

$$\begin{aligned}\mathrm{R}_2=\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}&T_\delta\big(u^\varepsilon(x)-u^\varepsilon(y)\big)\Big[\operatorname{div}\big(\sigma(y)\big)\big(\sigma(x)-\sigma(y)\big)(\nabla\varrho_\tau)(x-y)\\&-\sigma(y)\operatorname{div}\big(\sigma(y)\big)(\nabla\varrho_\tau)(x-y)-\sigma(y)\big(\sigma(x)-\sigma(y)\big)(\nabla^2\varrho_\tau)(x-y)\\&+\operatorname{div}\big(\sigma(y)\big)\operatorname{div}\big(\sigma(x)\big)\varrho_\tau(x-y)-\sigma(y)\operatorname{div}\big(\sigma(x)\big)(\nabla\varrho_\tau)(x-y)\Big]\mathrm{d}x\mathrm{d}y\mathrm{d}r\end{aligned}$$

hence

$$\begin{aligned}\mathrm{R}_1+\mathrm{R}_2=\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}&T_\delta\big(u^\varepsilon(x)-u^\varepsilon(y)\big)\\\times&\Big[2\big(\operatorname{div}\big(\sigma(x)\big)+\operatorname{div}\big(\sigma(y)\big)\big)\big(\sigma(x)-\sigma(y)\big)(\nabla\varrho_\tau)(x-y)\\&+\big(\sigma(x)-\sigma(y)\big)^2(\nabla^2\varrho_\tau)(x-y)+2\operatorname{div}\big(\sigma(x)\big)\operatorname{div}\big(\sigma(y)\big)\big)\varrho_\tau(x-y)\Big]\mathrm{d}x\mathrm{d}y\mathrm{d}r.\end{aligned}$$

Since $|T_\delta(\xi)|\leq|\xi|$, $\tau|\nabla\varrho_\tau|(\cdot)\leq C\varrho_{2\tau}(\cdot)$ and $\tau^2|\nabla^2\varrho_\tau|(\cdot)\leq C\varrho_{2\tau}(\cdot)$ with constants independent on τ, we deduce that

$$\mathrm{J}^\varepsilon\leq\mathrm{R}_1+\mathrm{R}_2\leq C_\sigma\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\varrho_{2\tau}(x-y)\big|u^\varepsilon(x)-u^\varepsilon(y)\big|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r$$

and therefore

$$\begin{aligned}&\mathbb{E}\int_{(\mathbb{T}^N)^2}\varrho_\tau(x-y)\big|u^\varepsilon(x,t)-u^\varepsilon(y,t)\big|\,\mathrm{d}x\,\mathrm{d}y\\&\leq\mathbb{E}\int_{(\mathbb{T}^N)^2}\varrho_\tau(x-y)\big|u_0(x)-u_0(y)\big|\,\mathrm{d}x\,\mathrm{d}y+C_T\big(\delta+\delta^{-1}\tau+\delta^{-1}\tau^2+\delta^\alpha\big)\\&\quad+C_\sigma\,\mathbb{E}\int_0^t\int_{(\mathbb{T}^N)^2}\varrho_{2\tau}(x-y)\big|u^\varepsilon(x,s)-u^\varepsilon(y,s)\big|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r.\end{aligned}$$

By optimization in δ, i.e. setting $\delta=\tau^\beta$, we obtain

$$\sup_{0<\tau<2D_N}\frac{C_T\big(\delta+\delta^{-1}\tau+\delta^{-1}\tau^2+\delta^\alpha\big)}{\tau^\varsigma}\leq C_T,$$

where the maximal choice of the parameter ς is $\min\big\{\frac{\alpha}{\alpha+1},\frac{1}{2}\big\}$ which corresponds to $\beta=\max\big\{\frac{1}{\alpha+1},\frac{1}{2}\big\}$. As a consequence,

$$\begin{aligned}&\mathbb{E}\int_{(\mathbb{T}^N)^2}\varrho_\tau(x-y)\big|u^\varepsilon(x,t)-u^\varepsilon(y,t)\big|\,\mathrm{d}x\,\mathrm{d}y\\&\leq C_T\bigg(\tau^\varsigma+\tau^\varsigma\,\mathbb{E}\,p^\varsigma(u_0)+\mathbb{E}\int_0^t\int_{\mathbb{T}^N}\varrho_{2\tau}(x-y)\big|u^\varepsilon(x,r)-u^\varepsilon(y,r)\big|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}r\bigg).\end{aligned}$$

Let us multiply the above by τ^{-1-s}, $s\in(0,\varsigma)$, and integrate with respect to $\tau\in(0,2D_N)$. As $|x-y|\leq\tau$ on the left hand side, we can estimate from below

$$\int_{|x-y|}^{2D_N}\frac{1}{\tau^{1+s}}\,\varrho_\tau(x-y)\mathrm{d}\tau=\frac{1}{|x-y|^{N+s}}\int_{|x-y|/2D_N}^1\lambda^{N+s-1}\varrho(\lambda)\mathrm{d}\lambda\geq\frac{C_s}{|x-y|^{N+s}}$$

and similarly for the last term on the right hand side we estimate from above

$$\int_{|x-y|/2}^{2D_N} \frac{1}{\tau^{1+s}}\,\varrho_{2\tau}(x-y)\,\mathrm{d}\tau \le \frac{C_s}{|x-y|^{N+s}}.$$

Accordingly,

$$\mathbb{E}\,p^s\big(u^\varepsilon(t)\big) \le C_{T,s}\bigg(1+\mathbb{E}\,p^\varsigma(u_0)+\mathbb{E}\int_0^t p^s\big(u^\varepsilon(r)\big)\,\mathrm{d}r\bigg)$$

and (3.34) follows by the Gronwall lemma. Furthermore, due to (3.31)

$$\mathbb{E}\|u^\varepsilon(t)\|_{L^1(\mathbb{T}^N)} \le \mathbb{E}\|u^\varepsilon(t)\|_{L^2(\mathbb{T}^N)} \le C\Big(1+\big(\mathbb{E}\|u_0\|^2_{L^2(\mathbb{T}^N)}\big)^{\frac{1}{2}}\Big)$$

hence we obtain (3.35). As a consequence of the previous estimate, the constant in (3.35) depends on the $L^2(\Omega;L^2(\mathbb{T}^N))$-norm of the initial condition. □

Corollary 3.4.5. *For all $\gamma\in(0,\varsigma)$ and $q>1$ satisfying $\gamma q<\sigma$, there exists a constant $C>0$ such that for all $\varepsilon\in(0,1)$*

$$\mathbb{E}\|u^\varepsilon\|^q_{L^q(0,T;W^{\gamma,q}(\mathbb{T}^N))} \le C. \tag{3.36}$$

Proof. The claim is a consequence of the bounds (3.31) and (3.35). Indeed, fix $\gamma\in(0,\varsigma)$ and $q\in(1,\infty)$. We will use an interpolation inequality:

$$\|\cdot\|_{W^{\gamma,q}(\mathbb{T}^N)} \le C\|\cdot\|^{1-\theta}_{W^{\gamma_0,q_0}(\mathbb{T}^N)}\|\cdot\|^{\theta}_{W^{\gamma_1,q_1}(\mathbb{T}^N)}, \tag{3.37}$$

where $\gamma_0,\gamma_1\in\mathbb{R}$, $q_0,q_1\in(0,\infty)$, $\gamma=(1-\theta)\gamma_0+\theta\gamma_1$, $\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$, $\theta\in(0,1)$, which follows from a more general result [75, Theorem 1.6.7] or [74, Theorem 2.4.1]. Fix $s\in(\gamma q,\varsigma)$ and set $\gamma_0=s$, $\gamma_1=0$, $q_0=1$, $q_1=p$. Then we obtain $\theta=\frac{s-\gamma}{s}$, $p=\frac{(s-\gamma)q}{s-\gamma q}$ and

$$\begin{aligned}
\mathbb{E}\int_0^T \|u^\varepsilon(t)\|^q_{W^{\gamma,q}(\mathbb{T}^N)}\,\mathrm{d}t &\le C\,\mathbb{E}\int_0^T\Big(\|u^\varepsilon(t)\|^{(1-\theta)q}_{W^{s,1}(\mathbb{T}^N)}\|u^\varepsilon(t)\|^{\theta q}_{L^p(\mathbb{T}^N)}\Big)\,\mathrm{d}t\\
&\le C\Big(\mathbb{E}\|u^\varepsilon(t)\|_{L^1(0,T;W^{s,1}(\mathbb{T}^N))}\Big)^{(1-\theta)q}\Big(\mathbb{E}\|u^\varepsilon(t)\|^p_{L^p(0,T;L^p(\mathbb{T}^N))}\Big)^{1-(1-\theta)q} \le C.
\end{aligned}$$

□

Also a better time regularity is needed.

Lemma 3.4.6. *Suppose that $\lambda\in(0,1/2)$, $q\in[2,\infty)$. There exists a constant $C>0$ such that for all $\varepsilon\in(0,1)$*

$$\mathbb{E}\|u^\varepsilon\|^q_{C^\lambda([0,T];H^{-2}(\mathbb{T}^N))} \le C. \tag{3.38}$$

Proof. Let $q\in[2,\infty)$. Recall that the set $\{u^\varepsilon;\,\varepsilon\in(0,1)\}$ is bounded in

$$L^q(\Omega;C(0,T;L^q(\mathbb{T}^N))).$$

Since all B^ε have the same polynomial growth we conclude, in particular, that

$$\{\mathrm{div}(B^\varepsilon(u^\varepsilon))\},\quad \{\mathrm{div}(A(x)\nabla u^\varepsilon)\},\quad \{\varepsilon\Delta u^\varepsilon\}$$

are bounded in $L^q(\Omega; C(0,T; H^{-2}(\mathbb{T}^N)))$ and consequently

$$\mathbb{E}\Big\| u^\varepsilon - \int_0^{\cdot} \varPhi^\varepsilon(u^\varepsilon)\,\mathrm{d}W\Big\|^q_{C^1([0,T];H^{-2}(\mathbb{T}^N))} \leq C.$$

In order to deal with the stochastic integral, let us recall the definition of the Riemann-Liouville operator: let X be a Banach space, $p \in (1,\infty]$, $\alpha \in (1/p, 1]$ and $f \in L^p(0,T;X)$, then we define

$$\big(R_\alpha f\big)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \qquad t \in [0,T].$$

It is well known that R_α is a bounded linear operator from $L^p(0,T;X)$ to the space of Hölder continuous functions $C^{\alpha-1/p}([0,T];X)$ (see e.g. [69, Theorem 3.6]). Assume now that $q \in (2,\infty)$, $\alpha \in (1/q, 1/2)$. Then according to the stochastic Fubini theorem [15, Theorem 4.18]

$$\int_0^t \varPhi^\varepsilon\big(u^\varepsilon(s)\big)\,\mathrm{d}W(s) = \big(R_\alpha Z\big)(t),$$

where

$$Z(s) = \frac{1}{\Gamma(1-\alpha)}\int_0^s (s-r)^{-\alpha}\,\varPhi^\varepsilon\big(u^\varepsilon(r)\big)\,\mathrm{d}W(r).$$

Therefore using the Burkholder-Davis-Gundy and Young inequality and the estimate (3.2)

$$\begin{aligned}
\mathbb{E}\Big\|\int_0^{\cdot} \varPhi^\varepsilon(u^\varepsilon)\,\mathrm{d}W\Big\|^q_{C^{\alpha-1/q}([0,T];L^2(\mathbb{T}^N))} &\leq C\,\mathbb{E}\|Z\|^q_{L^q(0,T;L^2(\mathbb{T}^N))}\\
&\leq C\int_0^T \mathbb{E}\bigg(\int_0^t \frac{1}{(t-s)^{2\alpha}}\|\varPhi^\varepsilon(u^\varepsilon)\|^2_{L_2(\mathfrak{U};L^2(\mathbb{T}^N))}\mathrm{d}s\bigg)^{\frac{q}{2}}\mathrm{d}t\\
&\leq CT^{\frac{q}{2}(1-2\alpha)}\mathbb{E}\int_0^T \Big(1+\|u^\varepsilon(s)\|^q_{L^2(\mathbb{T}^N)}\Big)\mathrm{d}s\\
&\leq CT^{\frac{q}{2}(1-2\alpha)}\Big(1+\|u^\varepsilon\|^q_{L^q(\Omega;L^q(0,T;L^2(\mathbb{T}^N)))}\Big) \leq C
\end{aligned}$$

and the claim follows. □

Corollary 3.4.7. *For all $\vartheta > 0$ there exist $\beta > 0$ and $C > 0$ such that for all $\varepsilon \in (0,1)$*

$$\mathbb{E}\|u^\varepsilon\|_{C^\beta([0,T];H^{-\vartheta}(\mathbb{T}^N))} \leq C. \tag{3.39}$$

Proof. If $\vartheta > 2$, the claim follows easily from (3.38) by the choice $\beta = \lambda$. If $\vartheta \in (0,2)$ the proof follows easily from interpolation between $H^{-2}(\mathbb{T}^N)$ and $L^2(\mathbb{T}^N)$. Indeed,

$$\begin{aligned}
\mathbb{E}\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|_{H^{-\vartheta}(\mathbb{T}^N)} &\leq C\,\mathbb{E}\Big(\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^{1-\theta}_{H^{-2}(\mathbb{T}^N)}\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^{\theta}_{L^2(\mathbb{T}^N)}\Big)\\
&\leq C\bigg(\mathbb{E}\Big(\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^{1-\theta}_{H^{-2}(\mathbb{T}^N)}\Big)^p\bigg)^{\frac{1}{p}}\bigg(\mathbb{E}\Big(\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^{\theta}_{L^2(\mathbb{T}^N)}\Big)^q\bigg)^{\frac{1}{q}}\\
&\leq C\bigg(1+\mathbb{E}\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^{(1-\theta)p}_{H^{-2}(\mathbb{T}^N)}\bigg)^{\frac{1}{p}}\bigg(1+\mathbb{E}\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^{\theta q}_{L^2(\mathbb{T}^N)}\bigg)^{\frac{1}{q}}
\end{aligned}$$

where the exponent p for the Hölder inequality is chosen in order to satisfy $(1-\theta)p=1$, i.e. since $\theta=\frac{2-\vartheta}{2}$, we have $p=\frac{2}{\vartheta}$. The first parenthesis can be estimated using (3.38) while the second one using (3.31). Similar computations yield the second part of the norm of $C^\beta([0,T];H^{-\vartheta}(\mathbb{T}^N))$. Indeed,

$$\begin{aligned}
&\mathbb{E}\sup_{\substack{0\leq s,t\leq T\\ s\neq t}}\frac{\|u^\varepsilon(t)-u^\varepsilon(s)\|_{H^{-\vartheta}(\mathbb{T}^N)}}{|t-s|^\beta}\\
&\quad\leq C\,\mathbb{E}\Bigg(\sup_{\substack{0\leq s,t\leq T\\ s\neq t}}\frac{\|u^\varepsilon(t)-u^\varepsilon(s)\|^{1-\theta}_{H^{-2}(\mathbb{T}^N)}}{|t-s|^\beta}\sup_{\substack{0\leq s,t\leq T\\ s\neq t}}\|u^\varepsilon(t)-u^\varepsilon(s)\|^\theta_{L^2(\mathbb{T}^N)}\Bigg)\\
&\quad\leq C\Bigg(1+\mathbb{E}\sup_{\substack{0\leq s,t\leq T\\ s\neq t}}\frac{\|u^\varepsilon(t)-u^\varepsilon(s)\|^{(1-\theta)p}_{H^{-2}(\mathbb{T}^N)}}{|t-s|^{\beta p}}\Bigg)^{\frac{1}{p}}\Bigg(1+\mathbb{E}\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^{\theta q}_{L^2(\mathbb{T}^N)}\Bigg)^{\frac{1}{q}}
\end{aligned}$$

where the same choice $p=\frac{2}{\vartheta}$ and the condition $\beta p\in(0,\frac{1}{2})$, which is needed for (3.38), gives (3.39) for $\beta\in(0,\frac{\vartheta}{4})$. □

Corollary 3.4.8. *Suppose that $\kappa\in(0,\frac{\varsigma}{2(4+\varsigma)})$. There exists a constant $C>0$ such that for all $\varepsilon\in(0,1)$*

$$\mathbb{E}\|u^\varepsilon\|_{H^\kappa(0,T;L^2(\mathbb{T}^N))}\leq C. \tag{3.40}$$

Proof. It follows from Lemma 3.4.6 that

$$\mathbb{E}\|u^\varepsilon\|^q_{H^\lambda(0,T;H^{-2}(\mathbb{T}^N))}\leq C, \tag{3.41}$$

where $\lambda\in(0,1/2)$, $q\in[1,\infty)$. Let $\gamma\in(0,\varsigma/2)$. If $\kappa=\theta\lambda$ and $0=-2\theta+(1-\theta)\gamma$ then it follows by the interpolation (see [2, Theorem 3.1]) and the Hölder inequality

$$\begin{aligned}
\mathbb{E}\|u^\varepsilon\|_{H^\kappa(0,T;L^2(\mathbb{T}^N))}&\leq C\,\mathbb{E}\Big(\|u^\varepsilon\|^\theta_{H^\lambda(0,T;H^{-2}(\mathbb{T}^N))}\|u^\varepsilon\|^{1-\theta}_{L^2(0,T;H^\gamma(\mathbb{T}^N))}\Big)\\
&\leq C\Big(\mathbb{E}\|u^\varepsilon\|^{\theta p}_{H^\lambda(0,T;H^{-2}(\mathbb{T}^N))}\Big)^{\frac{1}{p}}\Big(\mathbb{E}\|u^\varepsilon\|^{(1-\theta)r}_{L^2(0,T;H^\gamma(\mathbb{T}^N))}\Big)^{\frac{1}{r}},
\end{aligned}$$

where the exponent r is chosen in order to satisfy $(1-\theta)r=2$. The proof now follows from (3.36) and (3.41). □

Now, we have all in hand to conclude our compactness argument by showing tightness of a certain collection of laws. First, let us introduce some notation which will be used later on. If E is a Banach space and $t\in[0,T]$, we consider the space of continuous E-valued functions and denote by $\boldsymbol{\varrho}_t$ the operator of restriction to the interval $[0,t]$. To be more precise, we define

$$\begin{aligned}
\boldsymbol{\varrho}_t:C([0,T];E)&\longrightarrow C([0,t];E)\\
k&\longmapsto k|_{[0,t]}.
\end{aligned}\tag{3.42}$$

Plainly, $\boldsymbol{\varrho}_t$ is a continuous mapping. Let us define the path space

$$\mathcal{X}_u=\Big\{u\in L^2\big(0,T;L^2(\mathbb{T}^N)\big)\cap C\big([0,T];H^{-1}(\mathbb{T}^N)\big);\ \boldsymbol{\varrho}_0u\in L^2(\mathbb{T}^N)\Big\}$$

equipped with the norm

$$\|\cdot\|_{\mathcal{X}_u}=\|\cdot\|_{L^2(0,T;L^2(\mathbb{T}^N))}+\|\cdot\|_{C([0,T];H^{-1}(\mathbb{T}^N))}+\|\boldsymbol{\varrho}_0\cdot\|_{L^2(\mathbb{T}^N)}.$$

Next, we set $\mathcal{X}_W = C([0,T];\mathfrak{U}_0)$ and $\mathcal{X} = \mathcal{X}_u \times \mathcal{X}_W$. Let μ_{u^ε} denote the law of u^ε on $\mathcal{X}_u$, $\varepsilon \in (0,1)$, and μ_W the law of W on $\mathcal{X}_W$. Their joint law on $\mathcal{X}$ is then denoted by μ^ε.

Theorem 3.4.9. *The set $\{\mu^\varepsilon;\, \varepsilon \in (0,1)\}$ is tight and therefore relatively weakly compact in $\mathcal{X}$.*

Proof. First, we employ an Aubin-Dubinskii type compact embedding theorem which, in our setting, reads (see [54] for a general exposition; the proof of the following version can be found in [22]):

$$L^2(0,T;H^\gamma(\mathbb{T}^N)) \cap H^\kappa(0,T;L^2(\mathbb{T}^N)) \overset{c}{\hookrightarrow} L^2(0,T;L^2(\mathbb{T}^N)).$$

For $R > 0$ we define the set

$$\begin{aligned} B_{1,R} = \{u \in L^2(0,T;H^\gamma(\mathbb{T}^N)) \cap H^\kappa(0,T;L^2(\mathbb{T}^N));& \\ \|u\|_{L^2(0,T;H^\gamma(\mathbb{T}^N))} + \|u\|_{H^\kappa(0,T;L^2(\mathbb{T}^N))} \leq R\}& \end{aligned}$$

which is thus relatively compact in $L^2(0,T;L^2(\mathbb{T}^N))$. Moreover, by (3.36) and (3.40)

$$\begin{aligned} \mu_{u^\varepsilon}\big(B_{1,R}^C\big) &\leq \mathbb{P}\Big(\|u^\varepsilon\|_{L^2(0,T;H^\gamma(\mathbb{T}^N))} > \frac{R}{2}\Big) + \mathbb{P}\Big(\|u^\varepsilon\|_{H^\kappa(0,T;L^2(\mathbb{T}^N))} > \frac{R}{2}\Big) \\ &\leq \frac{2}{R}\Big(\mathbb{E}\|u^\varepsilon\|_{L^2(0,T;H^\gamma(\mathbb{T}^N))} + \mathbb{E}\|u^\varepsilon\|_{H^\kappa(0,T;L^2(\mathbb{T}^N))}\Big) \leq \frac{C}{R}. \end{aligned}$$

In order to prove tightness in $C([0,T];H^{-1}(\mathbb{T}^N))$ we employ the compact embedding

$$C^\beta([0,T];H^{-\vartheta}(\mathbb{T}^N)) \overset{c}{\hookrightarrow} C^{\tilde\beta}([0,T];H^{-1}(\mathbb{T}^N)) \hookrightarrow C([0,T];H^{-1}(\mathbb{T}^N)),$$

where $\tilde\beta < \beta$, $0 < \vartheta < 1$. Define

$$B_{2,R} = \{u \in C^\beta([0,T];H^{-\vartheta}(\mathbb{T}^N));\, \|u\|_{C^\beta([0,T];H^{-\vartheta}(\mathbb{T}^N))} \leq R\}$$

then by (3.39)

$$\mu_{u^\varepsilon}\big(B_{2,R}^C\big) \leq \frac{1}{R}\mathbb{E}\|u^\varepsilon\|_{C^\beta([0,T];H^{-\vartheta}(\mathbb{T}^N))} \leq \frac{C}{R}.$$

Tightness for the initial value is guaranteed as well since $u^\varepsilon(0) = u_0$ is smooth. As a consequence, the set

$$B_R = \big\{u \in B_{1,R} \cap B_{2,R};\, \|\varrho_0 u\|_{H^1(\mathbb{T}^N)} \leq R\big\}$$

is relatively compact in $\mathcal{X}_u$ and if $\eta > 0$ is given then for some suitably chosen $R > 0$ it holds true

$$\mu_{u^\varepsilon}(B_R) \geq 1 - \eta,$$

we obtain the tightness of $\{\mu_{u^\varepsilon};\, \varepsilon \in (0,1)\}$. Since also the laws μ_0 and μ_W are tight as being Radon measures on the Polish spaces $\mathcal{X}_0$ and $\mathcal{X}_W$, respectively, we conclude that also the set of their joint laws $\{\mu^\varepsilon;\, \varepsilon \in (0,1)\}$ is tight and Prokhorov's theorem therefore implies that it is also relatively weakly compact. □

Passing to a weakly convergent subsequence $\mu^n = \mu^{\varepsilon_n}$ (and denoting by μ the limit law) we now apply the Skorokhod representation theorem to infer the following proposition.

Proposition 3.4.10. *There exists a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ with a sequence of $\mathcal{X}$-valued random variables $(\tilde{u}^n, \tilde{W}^n)$, $n \in \mathbb{N}$, and $(\tilde{u}, \tilde{W})$ such that*

(i) the laws of $(\tilde{u}^n, \tilde{W}^n)$ and $(\tilde{u}, \tilde{W})$ under $\tilde{\mathbb{P}}$ coincide with μ^n and μ, respectively,

(ii) $(\tilde{u}^n, \tilde{W}^n)$ converges $\tilde{\mathbb{P}}$-almost surely to $(\tilde{u}, \tilde{W})$ in the topology of $\mathcal{X}$,

Remark 3.4.11. Note, that we can assume without loss of generality that the σ-algebra $\tilde{\mathscr{F}}$ is countably generated. This fact will be used later on for the application of the Banach-Alaoglu theorem. It should be also noted that the energy estimates remain valid also for the candidate solution $\tilde{u}$. Indeed, for any $p \in [1, \infty)$, it follows

$$\begin{aligned}\tilde{\mathbb{E}} \operatorname*{ess\,sup}_{0\leq t\leq T} \|\tilde{u}(t)\|^p_{L^p(\mathbb{T}^N)} &\leq \liminf_{n\to\infty} \tilde{\mathbb{E}} \sup_{0\leq t\leq T} \|\tilde{u}^n(t)\|^p_{L^p(\mathbb{T}^N)}\\ &= \liminf_{n\to\infty} \mathbb{E} \sup_{0\leq t\leq T} \|u^n(t)\|^p_{L^p(\mathbb{T}^N)} \leq C.\end{aligned}$$

Finally, let us define a complete, right-continuous filtration $(\tilde{\mathscr{F}}_t)$ such that all the processes $\tilde{u}$, $\tilde{W}$, $\tilde{u}^n$, $n \in \mathbb{N}$, are $(\tilde{\mathscr{F}}_t)$-adapted, that is

$$\tilde{\mathscr{F}}_t = \bigcap_{s>t} \sigma\Big(\sigma\big(\boldsymbol{\varrho}_s \tilde{u}, \boldsymbol{\varrho}_s \tilde{W}, \boldsymbol{\varrho}_s \tilde{u}^n, n \in \mathbb{N}\big) \cup \big\{N \in \tilde{\mathscr{F}};\ \tilde{\mathbb{P}}(N) = 0\big\}\Big), \quad t \in [0, T].$$

Then $\tilde{u}$, $\tilde{u}^n$, $n \in \mathbb{N}$, are $(\tilde{\mathscr{F}}_t)$-predictable $H^{-1}(\mathbb{T}^N)$-valued processes since they have continuous trajectories. Furthermore, by the embeddings $L^p(\mathbb{T}^N) \hookrightarrow H^{-1}(\mathbb{T}^N)$, $p \in [2, \infty)$, and $L^2(\mathbb{T}^N) \hookrightarrow L^p(\mathbb{T}^N)$, $p \in [1, 2)$, we conclude that, for all $p \in [1, \infty)$,

$$\tilde{u},\ \tilde{u}^n \in L^p(\tilde{\Omega} \times [0, T], \tilde{\mathcal{P}}, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t; L^p(\mathbb{T}^N)), \qquad n \in \mathbb{N},$$

where $\tilde{\mathcal{P}}$ denotes the predictable σ-algebra associated to $(\tilde{\mathscr{F}}_t)_{t\geq 0}$.

3.4.4 Passage to the limit

In this paragraph we provide the technical details of the identification of the limit process with a kinetic solution. The technique performed here will be used also in the proof of existence of a pathwise kinetic solution.

Theorem 3.4.12. *The triple $\big((\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}}), \tilde{W}, \tilde{u}\big)$ is a martingale kinetic solution to the problem (3.1).*

Let us define functions

$$f^n = \mathbf{1}_{u^n>\xi} : \Omega \times \mathbb{T}^N \times [0, T] \times \mathbb{R} \longrightarrow \mathbb{R},$$

$$\tilde{f}^n = \mathbf{1}_{\tilde{u}^n>\xi},\ \tilde{f} = \mathbf{1}_{\tilde{u}>\xi} : \tilde{\Omega} \times \mathbb{T}^N \times [0, T] \times \mathbb{R} \longrightarrow \mathbb{R},$$

and measures

$$\begin{aligned}\mathrm{d}m^n(x, t, \xi) &= \mathrm{d}n^n_1(x, t, \xi) + \mathrm{d}n^n_2(x, t, \xi),\\ \mathrm{d}\tilde{m}^n(x, t, \xi) &= \mathrm{d}\tilde{n}^n_1(x, t, \xi) + \mathrm{d}\tilde{n}^n_2(x, t, \xi),\end{aligned}$$

where

$$\begin{aligned}
\mathrm{d}n_1^n(x,t,\xi) &= \big|\sigma(x)\nabla u^n\big|^2\,\mathrm{d}\delta_{u^n(x,t)}(\xi)\,\mathrm{d}x\,\mathrm{d}t,\\
\mathrm{d}n_2^n(x,t,\xi) &= \varepsilon_n\big|\nabla u^n\big|^2\,\mathrm{d}\delta_{u^n(x,t)}(\xi)\,\mathrm{d}x\,\mathrm{d}t,\\
\mathrm{d}\tilde{n}_1^n(x,t,\xi) &= \big|\sigma(x)\nabla \tilde{u}^n\big|^2\,\mathrm{d}\delta_{u^n(x,t)}(\xi)\,\mathrm{d}x\,\mathrm{d}t,\\
\mathrm{d}\tilde{n}_2^n(x,t,\xi) &= \varepsilon_n\big|\nabla \tilde{u}^n\big|^2\,\mathrm{d}\delta_{u^n(x,t)}(\xi)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Remark, that $\tilde{f}^n$ and $\tilde{f}$, Borel functions of $\tilde{u}^n$ and $\tilde{u}$, respectively, and ξ, are measurable with respect to $\tilde{\mathcal{P}}\otimes\mathcal{B}(\mathbb{T}^N)\otimes\mathcal{B}(\mathbb{R})$. Besides, all the above measures are well defined. Indeed, Theorem 3.4.1 implies, in particular, that $u^n\in C([0,T];H^1(\mathbb{T}^N))$, $\mathbb{P}$-a.s., with $C([0,T];H^1(\mathbb{T}^N))$ being a Borel subset of $\mathcal{X}_u$ since the embedding $C([0,T];H^1(\mathbb{T}^N))\hookrightarrow\mathcal{X}_u$ is continuous. Thus, it follows from Proposition 3.4.10 that $\tilde{u}^n\in C([0,T];H^1(\mathbb{T}^N))$, $\tilde{\mathbb{P}}$-a.s., consequently $\tilde{m}^n(\psi):\Omega\to\mathbb{R}$ is measurable and

$$\tilde{m}^n(\psi)\overset{d}{\sim}m^n(\psi),\qquad\forall\psi\in C_0(\mathbb{T}^N\times[0,T]\times\mathbb{R}).$$

Let $\mathcal{M}_b(\mathbb{T}^N\times[0,T]\times\mathbb{R})$ denote the space of bounded Borel measures on $\mathbb{T}^N\times[0,T]\times\mathbb{R}$ whose norm is given by the total variation of measures. It is the dual space to the space of all continuous functions vanishing at infinity $C_0(\mathbb{T}^N\times[0,T]\times\mathbb{R})$ equipped with the supremum norm. This space is separable, so the following duality holds for $q,q^*\in(1,\infty)$ being conjugate exponents (see [19, Theorem 8.20.3]):

$$L^q_w(\tilde{\Omega};\mathcal{M}_b(\mathbb{T}^N\times[0,T]\times\mathbb{R}))\simeq\big(L^{q^*}(\tilde{\Omega};C_0(\mathbb{T}^N\times[0,T]\times\mathbb{R}))\big)^*,$$

where the space on the left hand side contains all weak*-measurable mappings $n:\tilde{\Omega}\to\mathcal{M}_b(\mathbb{T}^N\times[0,T]\times\mathbb{R})$ such that

$$\tilde{\mathbb{E}}\|n\|^q_{\mathcal{M}_b}<\infty.$$

Lemma 3.4.13. *It holds true (up to subsequences)*

(i) there exists a set of full Lebesgue measure $\mathcal{D}\subset[0,T]$ which contains $t=0$ such that

$$\tilde{f}^n(t)\overset{w^*}{\longrightarrow}\tilde{f}(t)\quad\text{in}\quad L^\infty(\tilde{\Omega}\times\mathbb{T}^N\times\mathbb{R})\text{-weak}^*,\qquad\forall t\in\mathcal{D},$$

(ii) there exists a kinetic measure $\tilde{m}$ such that

$$\tilde{m}^n\overset{w^*}{\longrightarrow}\tilde{m}\quad\text{in}\quad L^2_w(\tilde{\Omega};\mathcal{M}_b(\mathbb{T}^N\times[0,T]\times\mathbb{R}))\text{-weak}^*.\tag{3.43}$$

Moreover, $\tilde{m}$ can be rewritten as $\tilde{n}_1+\tilde{n}_2$, where

$$\mathrm{d}\tilde{n}_1(x,t,\xi)=\big|\sigma(x)\nabla\tilde{u}\big|^2\,\mathrm{d}\delta_{\tilde{u}(x,t)}(\xi)\,\mathrm{d}x\,\mathrm{d}t$$

and $\tilde{n}_2$ is almost surely a nonnegative measure over $\mathbb{T}^N\times[0,T]\times\mathbb{R}$.

Proof. According to Proposition 3.4.10, there exists a set $\Sigma\subset\tilde{\Omega}\times\mathbb{T}^N\times[0,T]$ of full measure and a subsequence still denoted by $\{\tilde{u}^n;\,n\in\mathbb{N}\}$ such that $\tilde{u}^n(\omega,x,t)\to\tilde{u}(\omega,x,t)$ for all $(\omega,x,t)\in\Sigma$. We infer that

$$\mathbf{1}_{\tilde{u}^n(\omega,x,t)>\xi}\longrightarrow\mathbf{1}_{\tilde{u}(\omega,x,t)>\xi}\tag{3.44}$$

whenever

$$\big(\tilde{\mathbb{P}}\otimes\mathcal{L}_{\mathbb{T}^N}\otimes\mathcal{L}_{[0,T]}\big)\big\{(\omega,x,t)\in\Sigma;\,\tilde{u}(\omega,x,t)=\xi\big\}=0,$$

where by $\mathcal{L}_{\mathbb{T}^N}$, $\mathcal{L}_{[0,T]}$ we denoted the Lebesque measure on $\mathbb{T}^N$ and $[0,T]$, respectively. However, the set

$$D=\Big\{\xi\in\mathbb{R};\,\Big(\tilde{\mathbb{P}}\otimes\mathcal{L}_{\mathbb{T}^N}\otimes\mathcal{L}_{[0,T]}\Big)\big(\tilde{u}=\xi\big)>0\Big\}$$

is at most countable since we deal with finite measures. To obtain a contradiction, suppose that D is uncountable and denote

$$D_k=\Big\{\xi\in\mathbb{R};\,\Big(\tilde{\mathbb{P}}\otimes\mathcal{L}_{\mathbb{T}^N}\otimes\mathcal{L}_{[0,T]}\Big)\big(\tilde{u}=\xi\big)>\frac{1}{k}\Big\},\qquad k\in\mathbb{N}.$$

Then $D=\cup_{k\in\mathbb{N}}D_k$ is a countable union so there exists $k_0\in\mathbb{N}$ such that D_{k_0} is uncountable. Hence

$$\begin{aligned}\Big(\tilde{\mathbb{P}}\otimes\mathcal{L}_{\mathbb{T}^N}\otimes\mathcal{L}_{[0,T]}\Big)\big(\tilde{u}\in D\big)&\geq\Big(\tilde{\mathbb{P}}\otimes\mathcal{L}_{\mathbb{T}^N}\otimes\mathcal{L}_{[0,T]}\Big)\big(\tilde{u}\in D_{k_0}\big)\\&=\sum_{\xi\in D_{k_0}}\Big(\tilde{\mathbb{P}}\otimes\mathcal{L}_{\mathbb{T}^N}\otimes\mathcal{L}_{[0,T]}\Big)\big(\tilde{u}=\xi\big)>\sum_{\xi\in D_{k_0}}\frac{1}{k_0}=\infty\end{aligned}$$

and the desired contradiction follows. We conclude that the convergence in (3.44) holds true for a.e. (ω,x,t,ξ) and obtain by the dominated convergence theorem

$$\tilde{f}^n\overset{w^*}{\longrightarrow}\tilde{f}\quad\text{in}\quad L^\infty(\tilde{\Omega}\times\mathbb{T}^N\times[0,T]\times\mathbb{R})\text{-weak}^*\tag{3.45}$$

hence (i) follows for a subsequence and the convergence at $t=0$ follows by a similar approach.

As the next step we shall show that the set $\{\tilde{m}^n;\,n\in\mathbb{N}\}$ is bounded in

$$L^2_w(\tilde{\Omega};\mathcal{M}_b(\mathbb{T}^N\times[0,T]\times\mathbb{R})).$$

With regard to the computations used in proof of the energy inequality, we get from (3.32)

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^N}\big|\sigma(x)\nabla u^n\big|^2\mathrm{d}x\,\mathrm{d}t+\varepsilon_n\int_0^T\int_{\mathbb{T}^N}\big|\nabla u^n\big|^2\mathrm{d}x\,\mathrm{d}t&\leq C\|u_0\|^2_{L^2(\mathbb{T}^N)}\\+C\sum_{k\geq1}\int_0^T\int_{\mathbb{T}^N}u^n g_k^n(x,u^n)\,\mathrm{d}x\,\mathrm{d}\beta_k(t)&+C\int_0^T\int_{\mathbb{T}^N}G_n^2(x,u^n)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Taking square and expectation and finally by the Itô isometry, we deduce

$$\begin{aligned}\tilde{\mathbb{E}}\big|\tilde{m}^n(\mathbb{T}^N\times[0,T]\times\mathbb{R})\big|^2&=\mathbb{E}\big|m^n(\mathbb{T}^N\times[0,T]\times\mathbb{R})\big|^2\\&=\mathbb{E}\bigg|\int_0^T\int_{\mathbb{T}^N}\big|\sigma(x)\nabla u^n\big|^2\mathrm{d}x\,\mathrm{d}t+\varepsilon_n\int_0^T\int_{\mathbb{T}^N}\big|\nabla u^n\big|^2\mathrm{d}x\,\mathrm{d}t\bigg|^2\leq C.\end{aligned}$$

Thus, according to the Banach-Alaoglu theorem, (3.43) is obtained (up to subsequence). However, it still remains to show that the weak* limit $\tilde{m}$ is actually a kinetic measure. The first point of Definition 3.2.1 is straightforward as it corresponds to the weak*-measurability of $\tilde{m}$. The second one giving the behavior for large ξ follows from the uniform estimate (3.32). Indeed, let (χ_δ) be a truncation on $\mathbb{R}$, then it holds, for

$p\in[2,\infty)$,

$$\begin{aligned}\tilde{\mathbb{E}}\int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}}|\xi|^{p-2}\,\mathrm{d}\tilde{m}(x,t,\xi)&\leq\liminf_{\delta\to0}\tilde{\mathbb{E}}\int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}}|\xi|^{p-2}\chi_\delta(\xi)\,\mathrm{d}\tilde{m}(x,t,\xi)\\&=\liminf_{\delta\to0}\lim_{n\to\infty}\tilde{\mathbb{E}}\int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}}|\xi|^{p-2}\chi_\delta(\xi)\,\mathrm{d}\tilde{m}^n(x,t,\xi)\leq C,\end{aligned}$$

where the last inequality follows from (3.32) and the sequel. As a consequence, $\tilde{m}$ vanishes for large ξ. In order to verify the remaining requirement of Definition 3.2.1, let us define

$$x^n(t)=\int_{\mathbb{T}^N\times[0,t]\times\mathbb{R}}\psi(x,\xi)\,\mathrm{d}\tilde{m}^n(s,x,\xi)$$

and take the limit as $n\to\infty$. These processes are predictable due to the definition of measures $\tilde{m}^n$. Let $\alpha\in L^2(\tilde{\Omega})$, $\gamma\in L^2(0,T)$, then, by the Fubini theorem,

$$\tilde{\mathbb{E}}\bigg(\alpha\int_0^T\gamma(t)x^n(t)\,\mathrm{d}t\bigg)=\tilde{\mathbb{E}}\bigg(\alpha\int_{[0,T]\times\mathbb{T}^N\times\mathbb{R}}\psi(x,\xi)\Gamma(s)\,\mathrm{d}\tilde{m}^n(s,x,\xi)\bigg)$$

where $\Gamma(s)=\int_s^T\gamma(t)\,\mathrm{d}t$. Hence, since Γ is continuous, we obtain by the weak convergence of $\tilde{m}^n$ to $\tilde{m}$

$$\tilde{\mathbb{E}}\bigg(\alpha\int_0^T\gamma(t)x^n(t)\,\mathrm{d}t\bigg)\longrightarrow\tilde{\mathbb{E}}\bigg(\alpha\int_0^T\gamma(t)x(t)\,\mathrm{d}t\bigg),$$

where

$$x(t)=\int_{\mathbb{T}^N\times[0,t]\times\mathbb{R}}\psi(x,\xi)\,\mathrm{d}\tilde{m}(s,x,\xi).$$

Consequently, x^n converges to x weakly in $L^2(\tilde{\Omega}\times[0,T])$ and, in particular, since the space of predictable L^2-integrable functions is weakly closed, the claim follows.

Finally, by the same approach as above, we deduce that there exist kinetic measures $\tilde{o}_1$, $\tilde{o}_2$ such that

$$\tilde{n}_1^n\overset{w^*}{\longrightarrow}\tilde{o}_1,\quad\tilde{n}_2^n\overset{w^*}{\longrightarrow}\tilde{o}_2\qquad\text{in}\quad L^2_w(\tilde{\Omega};\mathcal{M}_b(\mathbb{T}^N\times[0,T]\times\mathbb{R}))\text{-weak}^*.$$

Then from (3.32) we obtain

$$\tilde{\mathbb{E}}\int_0^T\int_{\mathbb{T}^N}\big|\sigma(x)\nabla\tilde{u}^n\big|^2\mathrm{d}x\,\mathrm{d}t\leq C$$

hence application of the Banach-Alaoglu theorem yields that, up to subsequence, $\sigma\nabla\tilde{u}^n$ converges weakly in $L^2(\tilde{\Omega}\times\mathbb{T}^N\times[0,T])$. On the other hand, from the strong convergence given by Proposition 3.4.10 and the fact that $\sigma\in W^{1,\infty}(\mathbb{T}^N)$, we conclude using integration by parts, for all $\psi\in C^1(\mathbb{T}^N\times[0,T])$,

$$\int_0^T\int_{\mathbb{T}^N}\sigma(x)\nabla\tilde{u}^n\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t\longrightarrow\int_0^T\int_{\mathbb{T}^N}\sigma(x)\nabla\tilde{u}\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t,\quad\tilde{\mathbb{P}}\text{-a.s.}.$$

Therefore

$$\sigma\nabla\tilde{u}^n\overset{w}{\longrightarrow}\sigma\nabla\tilde{u},\quad\text{in}\ \ L^2(\mathbb{T}^N\times[0,T]),\quad\tilde{\mathbb{P}}\text{-a.s.}.$$

Since any norm is weakly sequentially lower semicontinuous, it follows for all $\varphi \in C_0(\mathbb{T}^N \times [0,T] \times \mathbb{R})$ and fixed $\xi \in \mathbb{R}$, $\tilde{\mathbb{P}}$-a.s.,

$$\int_0^T \int_{\mathbb{T}^N} \big|\sigma(x)\nabla\tilde{u}\big|^2 \varphi^2(x,t,\xi)\,\mathrm{d}x\,\mathrm{d}t \leq \liminf_{n\to\infty} \int_0^T \int_{\mathbb{T}^N} \big|\sigma(x)\nabla\tilde{u}^n\big|^2 \varphi^2(x,t,\xi)\mathrm{d}x\mathrm{d}t$$

and by the Fatou lemma

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} \big|\sigma(x)\nabla\tilde{u}\big|^2 \varphi^2(x,t,\xi)\,\mathrm{d}\delta_{\tilde{u}=\xi}\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \leq \liminf_{n\to\infty} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} \big|\sigma(x)\nabla\tilde{u}^n\big|^2 \varphi^2(x,t,\xi)\,\mathrm{d}\delta_{\tilde{u}^n=\xi}\,\mathrm{d}x\,\mathrm{d}t, \quad \tilde{\mathbb{P}}\text{-a.s..} \end{aligned}$$

In other words, this gives $\tilde{n}_1 = |\sigma\,\nabla\tilde{u}|^2\delta_{\tilde{u}=\xi} \leq \tilde{o}_1$ $\tilde{\mathbb{P}}$-a.s. hence $\tilde{n}_2 = \tilde{o}_2 + (\tilde{o}_1 - \tilde{n}_1)$ is $\tilde{\mathbb{P}}$-a.s. a nonnegative measure and the proof is complete. □

Note, that as the set $\mathcal{D}$ is a complement of a set with zero Lebesgue measure, it is dense in $[0,T]$. Let us define for all $t \in \mathcal{D}$ and some fixed $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$

$$\begin{aligned} M^n(t) = \big\langle f^n(t),\varphi\big\rangle - \big\langle f_0,\varphi\big\rangle &- \int_0^t \big\langle f^n(s), b^n(\xi)\cdot\nabla\varphi\big\rangle\,\mathrm{d}s \\ &- \int_0^t \Big\langle f^n(s), \operatorname{div}\big(A(x)\nabla\varphi\big)\Big\rangle\,\mathrm{d}s - \varepsilon_n \int_0^t \big\langle f^n(s),\Delta\varphi\big\rangle\,\mathrm{d}s \\ &- \frac{1}{2}\int_0^t \big\langle \delta_{u^n=\xi}G_n^2, \partial_\xi\varphi\big\rangle\,\mathrm{d}s + \big\langle m^n, \partial_\xi\varphi\big\rangle([0,t)), \qquad n \in \mathbb{N}, \end{aligned}$$

$$\begin{aligned} \tilde{M}^n(t) = \big\langle \tilde{f}^n(t),\varphi\big\rangle - \big\langle \tilde{f}^n(0),\varphi\big\rangle &- \int_0^t \big\langle \tilde{f}^n(s), b^n(\xi)\cdot\nabla\varphi\big\rangle\,\mathrm{d}s \\ &- \int_0^t \Big\langle \tilde{f}^n(s), \operatorname{div}\big(A(x)\nabla\varphi\big)\Big\rangle\,\mathrm{d}s - \varepsilon_n \int_0^t \big\langle \tilde{f}^n(s),\Delta\varphi\big\rangle\,\mathrm{d}s \\ &- \frac{1}{2}\int_0^t \big\langle \delta_{\tilde{u}^n=\xi}G_n^2, \partial_\xi\varphi\big\rangle\,\mathrm{d}s + \big\langle \tilde{m}^n, \partial_\xi\varphi\big\rangle([0,t)), \qquad n \in \mathbb{N}, \end{aligned}$$

$$\begin{aligned} \tilde{M}(t) = \big\langle \tilde{f}(t),\varphi\big\rangle - \big\langle \tilde{f}(0),\varphi\big\rangle &- \int_0^t \big\langle \tilde{f}(s), b(\xi)\cdot\nabla\varphi\big\rangle\,\mathrm{d}s \\ &- \int_0^t \Big\langle \tilde{f}(s), \operatorname{div}\big(A(x)\nabla\varphi\big)\Big\rangle\,\mathrm{d}s - \frac{1}{2}\int_0^t \big\langle \delta_{\tilde{u}=\xi}G^2, \partial_\xi\varphi\big\rangle\,\mathrm{d}s \\ &+ \big\langle \tilde{m}, \partial_\xi\varphi\big\rangle([0,t)). \end{aligned}$$

The proof of Theorem 3.4.12 is a consequence of the following two propositions.

Proposition 3.4.14. *The process $\tilde{W}$ is a $(\tilde{\mathscr{F}}_t)$-cylindrical Wiener process, i.e. there exists a collection of mutually independent real-valued $(\tilde{\mathscr{F}}_t)$-Wiener processes $\{\tilde{\beta}_k\}_{k\geq 1}$ such that $\tilde{W} = \sum_{k\geq 1}\tilde{\beta}_k e_k$.*

Proof. Hereafter, times $s,t \in [0,T]$, $s \leq t$, and a continuous function

$$\gamma : C\big([0,s]; H^{-1}(\mathbb{T}^N)\big) \times C\big([0,s];\mathfrak{U}_0\big) \longrightarrow [0,1]$$

will be fixed but otherwise arbitrary and by $\boldsymbol{\varrho}_s$ we denote the operator of restriction to the interval $[0,s]$ as introduced in (3.42).

Obviously, $\tilde W$ is a $\mathfrak{U}_0$-valued cylindrical Wiener process and is $(\tilde{\mathscr{F}}_t)$-adapted. According to the Lévy martingale characterization theorem, it remains to show that it is also a $(\tilde{\mathscr{F}}_t)$-martingale. It holds true

$$\tilde{\mathbb{E}}\,\gamma\big(\boldsymbol{\varrho}_s\tilde u^n,\boldsymbol{\varrho}_s\tilde W^n\big)\big[\tilde W^n(t)-\tilde W^n(s)\big]=\mathbb{E}\,\gamma\big(\boldsymbol{\varrho}_s u^n,\boldsymbol{\varrho}_s W\big)\big[W(t)-W(s)\big]=0$$

since W is a martingale and the laws of $(\tilde u^n,\tilde W^n)$ and (u^n,W) coincide. Next, the uniform estimate

$$\sup_{n\in\mathbb{N}}\tilde{\mathbb{E}}\|\tilde W^n(t)\|^2_{\mathfrak{U}_0}=\sup_{n\in\mathbb{N}}\mathbb{E}\|W(t)\|^2_{\mathfrak{U}_0}<\infty$$

and the Vitali convergence theorem yields

$$\tilde{\mathbb{E}}\,\gamma\big(\boldsymbol{\varrho}_s\tilde u,\boldsymbol{\varrho}_s\tilde W\big)\big[\tilde W(t)-\tilde W(s)\big]=0$$

which finishes the proof. □

Proposition 3.4.15. *The processes*

$$\tilde M(t),\quad \tilde M^2(t)-\sum_{k\geq1}\int_0^t\big\langle\delta_{\tilde u=\xi}\,g_k,\varphi\big\rangle^2\,\mathrm{d}r,\quad \tilde M(t)\tilde\beta_k(t)-\int_0^t\big\langle\delta_{\tilde u=\xi}\,g_k,\varphi\big\rangle\,\mathrm{d}r,$$

indexed by $t\in\mathcal{D}$, are $(\tilde{\mathscr{F}}_t)$-martingales.

Proof. All these processes are $(\tilde{\mathscr{F}}_t)$-adapted as they are Borel functions of $\tilde u$ and $\tilde\beta_k$, $k\in\mathbb{N}$, up to time t. For the rest, we use the same approach and notation as the one used in the previous lemma. Let us denote by $\tilde\beta_k^n$, $k\geq1$ the real-valued Wiener processes corresponding to $\tilde W^n$, that is $\tilde W^n=\sum_{k\geq1}\tilde\beta_k^n e_k$. For all $n\in\mathbb{N}$, the process

$$M^n=\int_0^{\cdot}\big\langle\delta_{u^n=\xi}\,\varPhi^n(u^n)\mathrm{d}W,\varphi\big\rangle=\sum_{k\geq1}\int_0^{\cdot}\big\langle\delta_{u^n=\xi}\,g_k^n,\varphi\big\rangle\,\mathrm{d}\beta_k(r)$$

is a square integrable $(\mathscr{F}_t)$-martingale by (3.2) and by the fact that the set $\{u^n;\ n\in\mathbb{N}\}$ is bounded in $L^2(\Omega;L^2(0,T;L^2(\mathbb{T}^N)))$. Therefore

$$(M^n)^2-\sum_{k\geq1}\int_0^{\cdot}\big\langle\delta_{u^n=\xi}\,g_k^n,\varphi\big\rangle^2\,\mathrm{d}r,\qquad M^n\beta_k-\int_0^{\cdot}\big\langle\delta_{u^n=\xi}\,g_k^n,\varphi\big\rangle\,\mathrm{d}r$$

are $(\mathscr{F}_t)$-martingales and this implies together with the equality of laws

$$\tilde{\mathbb{E}}\,\gamma\big(\boldsymbol{\varrho}_s\tilde u^n,\boldsymbol{\varrho}_s\tilde W^n\big)\big[\tilde M^n(t)-\tilde M^n(s)\big]=\mathbb{E}\,\gamma\big(\boldsymbol{\varrho}_s u^n,\boldsymbol{\varrho}_s W\big)\big[M^n(t)-M^n(s)\big]=0,\tag{3.46}$$

$$\begin{aligned}&\tilde{\mathbb{E}}\,\gamma\big(\boldsymbol{\varrho}_s\tilde u^n,\boldsymbol{\varrho}_s\tilde W^n\big)\Big[(\tilde M^n)^2(t)-(\tilde M^n)^2(s)-\sum_{k\geq1}\int_s^t\big\langle\delta_{\tilde u^n=\xi}\,g_k^n,\varphi\big\rangle^2\,\mathrm{d}r\Big]\\&=\mathbb{E}\,\gamma\big(\boldsymbol{\varrho}_s u^n,\boldsymbol{\varrho}_s W\big)\Big[(M^n)^2(t)-(M^n)^2(s)-\sum_{k\geq1}\int_s^t\big\langle\delta_{u^n=\xi}\,g_k^n,\varphi\big\rangle^2\,\mathrm{d}r\Big]=0,\end{aligned}\tag{3.47}$$

$$\begin{aligned}&\tilde{\mathbb{E}}\,\gamma\big(\boldsymbol{\varrho}_s\tilde u^n,\boldsymbol{\varrho}_s\tilde W^n\big)\Big[\tilde M^n(t)\tilde\beta_k^n(t)-\tilde M^n(s)\tilde\beta_k^n(s)-\int_s^t\big\langle\delta_{\tilde u^n=\xi}\,g_k^n,\varphi\big\rangle\,\mathrm{d}r\Big]\\&=\mathbb{E}\,\gamma\big(\boldsymbol{\varrho}_s u^n,\boldsymbol{\varrho}_s W\big)\Big[M^n(t)\beta_k(t)-M^n(s)\beta_k(s)-\int_s^t\big\langle\delta_{u^n=\xi}\,g_k^n,\varphi\big\rangle\,\mathrm{d}r\Big]=0.\end{aligned}\tag{3.48}$$

Moreover, for any $s,\,t\in\mathcal{D}$, $s\le t$, the expectations in (3.46)-(3.48) converge by the Vitali convergence theorem. Indeed, all terms are uniformly integrable by (3.2) and (3.31) and converge $\tilde{\mathbb{P}}$-a.s. (after extracting a subsequence) due to Lemma 3.4.13, (3.44), (3.45), Proposition 3.4.10 and the construction of Φ^ε, B^ε. Hence

$$\begin{aligned}
&\tilde{\mathbb{E}}\,\gamma\big(\boldsymbol{\varrho}_s\tilde u,\boldsymbol{\varrho}_s\tilde W\big)\big[\tilde M(t)-\tilde M(s)\big]=0,\\
&\tilde{\mathbb{E}}\,\gamma\big(\boldsymbol{\varrho}_s\tilde u,\boldsymbol{\varrho}_s\tilde W\big)\bigg[\tilde M^2(t)-\tilde M^2(s)-\sum_{k\ge1}\int_s^t\big\langle\delta_{\tilde u=\xi}\,g_k,\varphi\big\rangle^2\mathrm{d}r\bigg]=0,\\
&\tilde{\mathbb{E}}\,\gamma\big(\boldsymbol{\varrho}_s\tilde u,\boldsymbol{\varrho}_s\tilde W\big)\bigg[\tilde M(t)\tilde\beta_k(t)-\tilde M(s)\tilde\beta_k(s)-\int_s^t\big\langle\delta_{\tilde u=\xi}\,g_k,\varphi\big\rangle\mathrm{d}r\bigg]=0,
\end{aligned}$$

which gives the $(\tilde{\mathscr{F}}_t)$-martingale property. $\square$

Proof of Theorem 3.4.12. If all the processes in 3.4.15 were continuous-time martingales then it would hold true

$$\bigg\langle\!\!\bigg\langle\tilde M-\int_0^{\cdot}\big\langle\delta_{\tilde u=\xi}\,\Phi(\tilde u)\,\mathrm{d}\tilde W,\varphi\big\rangle\bigg\rangle\!\!\bigg\rangle=0,$$

where by $\langle\!\langle\cdot\rangle\!\rangle$ we denote the quadratic variation process, and therefore, for every $\varphi\in C_c^\infty(\mathbb{T}^N\times\mathbb{R})$, $t\in[0,T]$, $\tilde{\mathbb{P}}$-a.s.,

$$\begin{aligned}
\big\langle\tilde f(t),\varphi\big\rangle-\big\langle\tilde f_0,\varphi\big\rangle-\int_0^t\big\langle\tilde f(s),b(\xi)\cdot\nabla\varphi\big\rangle\mathrm{d}s-\int_0^t\big\langle\tilde f(s),\operatorname{div}\big(A(x)\nabla\varphi\big)\big\rangle\mathrm{d}s\\
=\int_0^t\big\langle\delta_{\tilde u=\xi}\,\Phi(\tilde u)\,\mathrm{d}\tilde W,\varphi\big\rangle+\frac12\int_0^t\big\langle\delta_{\tilde u=\xi}G^2,\partial_\xi\varphi\big\rangle\,\mathrm{d}s-\big\langle\tilde m,\partial_\xi\varphi\big\rangle([0,t))
\end{aligned}\tag{3.49}$$

and the proof would be completed with $\tilde u$ satisfying the kinetic formulation even in a stronger sense than required by Definition 3.2.2.

In the case of martingales indexed by $t\in\mathcal{D}$, we employ Proposition 3.A.1 to conclude the validity of (3.49) for all $\varphi\in C_c^\infty(\mathbb{T}^N\times\mathbb{R})$, $t\in\mathcal{D}$, $\tilde{\mathbb{P}}$-a.s., and we need to allow a formulation which is weak also in time. Mimicking the technique developed in order to derive the kinetic formulation in Section 3.2, let us define

$$\begin{aligned}
N(t)=\big\langle\tilde f_0,\varphi\big\rangle+\int_0^t\big\langle\tilde f(s),b(\xi)\cdot\nabla\varphi\big\rangle\mathrm{d}s+\int_0^t\big\langle\tilde f(s),\operatorname{div}\big(A(x)\nabla\varphi\big)\big\rangle\mathrm{d}s\\
+\int_0^t\big\langle\delta_{\tilde u=\xi}\,\Phi(\tilde u)\,\mathrm{d}\tilde W,\varphi\big\rangle+\frac12\int_0^t\big\langle\delta_{\tilde u=\xi}G^2,\partial_\xi\varphi\big\rangle\,\mathrm{d}s.
\end{aligned}$$

Note, that N is a continuous real-valued semimartingale and

$$N(t)=\big\langle\tilde f(t),\varphi\big\rangle+\big\langle\tilde m,\partial_\xi\varphi\big\rangle([0,t)),\qquad\forall t\in\mathcal{D}.$$

Next, we apply the Itô formula to calculate the stochastic differential of the product $N(t)\varphi_1(t)$, where $\varphi_1\in C_c^\infty([0,T))$. After application of the Fubini theorem to the term including the kinetic measure $\tilde m$, we obtain exactly the formulation (3.8). $\square$

3.4.5 Pathwise solutions

In order to finish the proof, we make use of the Gyöngy-Krylov characterization of convergence in probability introduced in [29]. It is useful in situations when the pathwise uniqueness and the existence of at least one martingale solution imply the existence of a unique pathwise solution.

Proposition 3.4.16. *Let X be a Polish space equipped with the Borel σ-algebra. A sequence of X-valued random variables $\{Y_n;\, n\in\mathbb{N}\}$ converges in probability if and only if for every subsequence of joint laws, $\{\mu_{n_k,m_k};\, k\in\mathbb{N}\}$, there exists a further subsequence which converges weakly to a probability measure μ such that*

$$\mu\big((x,y)\in X\times X;\, x=y\big)=1.$$

We consider the collection of joint laws of (u^n,u^m) on $\mathcal{X}_u\times\mathcal{X}_u$, denoted by $\mu_u^{n,m}$. For this purpose we define the extended path space

$$\mathcal{X}^J=\mathcal{X}_u\times\mathcal{X}_u\times\mathcal{X}_W$$

As above, denote by μ_W the law of W and set $\nu^{n,m}$ to be the joint law of (u^n,u^m,W). Similarly to Proposition 3.4.9 the following fact holds true. The proof is nearly identical and so will be left to the reader.

Proposition 3.4.17. *The collection $\{\nu^{n,m};\, n,m\in\mathbb{N}\}$ is tight on $\mathcal{X}^J$.*

Let us take any subsequence $\{\nu^{n_k,m_k};\, k\in\mathbb{N}\}$. By the Prokhorov theorem, it is relatively weakly compact hence it contains a weakly convergent subsequence. Without loss of generality we may assume that the original sequence $\{\nu^{n_k,m_k};\, k\in\mathbb{N}\}$ itself converges weakly to a measure ν. According to the Skorokhod representation theorem, we infer the existence of a probability space $(\bar\Omega,\bar{\mathscr{F}},\bar{\mathbb{P}})$ with a sequence of random variables $(\hat u^{n_k},\check u^{m_k},\bar W^k), k\in\mathbb{N}$, converging almost surely in $\mathcal{X}^J$ to a random variable $(\hat u,\check u,\bar W)$ and

$$\bar{\mathbb{P}}\big((\hat u^{n_k},\check u^{m_k},\bar W^k)\in\cdot\,\big)=\nu^{n_k,m_k}(\cdot),\qquad \bar{\mathbb{P}}\big((\hat u,\check u,\bar W)\in\cdot\,\big)=\nu(\cdot).$$

Observe that in particular, $\mu_u^{n_k,m_k}$ converges weakly to a measure μ_u defined by

$$\mu_u(\cdot)=\bar{\mathbb{P}}\big((\hat u,\check u)\in\cdot\,\big).$$

As the next step, we should recall the technique established in the previous section. Analogously, it can be applied to both $(\hat u^{n_k},\bar W^k)$, $(\hat u,\bar W)$ and $(\check u^{m_k},\bar W^k)$, $(\check u,\bar W)$ in order to show that $(\hat u,\bar W)$ and $(\check u,\bar W)$ are martingale kinetic solutions of (3.1) defined on the same stochastic basis $(\bar\Omega,\bar{\mathscr{F}},(\bar{\mathscr{F}}_t),\bar{\mathbb{P}})$, where

$$\bar{\mathscr{F}}_t=\sigma\big(\sigma\big(\boldsymbol{\varrho}_t\hat u,\,\boldsymbol{\varrho}_t\check u,\,\boldsymbol{\varrho}_t\bar W\big)\cup\big\{N\in\bar{\mathscr{F}};\,\bar{\mathbb{P}}(N)=0\big\}\big),\quad t\in[0,T].$$

Since $\hat u(0)=\check u(0)=\bar u_0$, $\bar{\mathbb{P}}$-a.s., we infer from Theorem 3.3.3 that $\hat u=\check u$ in $\mathcal{X}_u$, $\bar{\mathbb{P}}$-a.s., hence

$$\mu_u\big((x,y)\in\mathcal{X}_u\times\mathcal{X}_u;\, x=y\big)=\bar{\mathbb{P}}\big(\hat u=\check u\text{ in }\mathcal{X}_u\big)=1.$$

Now, we have all in hand to apply Proposition 3.4.16. It implies that the original sequence u^n defined on the initial probability space $(\Omega,\mathscr{F},\mathbb{P})$ converges in probability in the topology of $\mathcal{X}_u$ to a random variable u. Without loss of generality, we assume that u^n converges to u almost surely in $\mathcal{X}_u$ and again by the method from Section 3.4.4 we finally deduce that u is a pathwise kinetic solution to (3.1). Actually, identification of the limit is more straightforward here since in this case all the work is done for the initial setting and only one fixed driving Wiener process W is considered.

3.5 Existence - general initial data

In this final section we provide an existence proof in the general case of

$$u_0 \in L^p(\Omega; L^p(\mathbb{T}^N)), \quad \forall p \in [1, \infty).$$

It is a straightforward consequence of the previous section. We approximate the initial condition by a sequence $\{u_0^\varepsilon\} \subset L^p(\Omega; C^\infty(\mathbb{T}^N))$, $p \in [1, \infty)$, such that $u_0^\varepsilon \to u_0$ in $L^1(\Omega; L^1(\mathbb{T}^N))$. That is, the initial condition u_0^ε can be defined as a pathwise mollification of u_0 so that it holds true

$$\|u_0^\varepsilon\|_{L^p(\Omega; L^p(\mathbb{T}^N))} \leq \|u_0\|_{L^p(\Omega; L^p(\mathbb{T}^N))}, \qquad \varepsilon \in (0, 1),\ p \in [1, \infty). \tag{3.50}$$

According to the previous section, for each $\varepsilon \in (0, 1)$, there exists a kinetic solution u^ε to (3.1) with initial condition u_0^ε. By application of the comparison principle (3.23),

$$\mathbb{E}\|u^{\varepsilon_1}(t) - u^{\varepsilon_2}(t)\|_{L^1(\mathbb{T}^N)} \leq \mathbb{E}\|u_0^{\varepsilon_1} - u_0^{\varepsilon_2}\|_{L^1(\mathbb{T}^N)}, \qquad \varepsilon_1, \varepsilon_2 \in (0, 1),$$

hence $\{u^\varepsilon;\, \varepsilon \in (0, 1)\}$ is a Cauchy sequence in $L^1(\Omega \times [0, T], \mathcal{P}, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t; L^1(\mathbb{T}^N))$. Consequently, there exists $u \in L^1(\Omega \times [0, T], \mathcal{P}, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t; L^1(\mathbb{T}^N))$ such that

$$u^\varepsilon \longrightarrow u \quad \text{in} \quad L^1(\Omega \times [0, T], \mathcal{P}, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t; L^1(\mathbb{T}^N)).$$

By (3.50) and Remark 3.4.11, we still have the uniform energy estimates, $p \in [1, \infty)$,

$$\mathbb{E} \operatorname*{ess\,sup}_{0 \leq t \leq T} \|u^\varepsilon(t)\|^p_{L^p(\mathbb{T}^N)} \leq C_{T, u_0}. \tag{3.51}$$

as well as (using the usual notation)

$$\mathbb{E}\big|m^\varepsilon(\mathbb{T}^N \times [0, T] \times \mathbb{R})\big|^2 \leq C_{T, u_0}.$$

Thus, using this observations as in Lemma 3.4.13, one finds that there exists a subsequence $\{u^n;\, n \in \mathbb{N}\}$ such that

(i) $f^n \xrightarrow{w^*} f \quad$ in $\quad L^\infty(\Omega \times \mathbb{T}^N \times [0, T] \times \mathbb{R})$-weak*,

(ii) there exists a kinetic measure m such that

$$m^n \xrightarrow{w^*} m \quad \text{in} \quad L^2_w(\Omega; \mathcal{M}_b(\mathbb{T}^N \times [0, T] \times \mathbb{R}))\text{-weak}^*$$

and $m = n_1 + n_2$, where

$$\mathrm{d}n_1(x, t, \xi) = \big|\sigma(x)\nabla u\big|^2\, \mathrm{d}\delta_{u(x,t)}(\xi)\, \mathrm{d}x\, \mathrm{d}t$$

and n_2 is almost surely a nonnegative measure over $\mathbb{T}^N \times [0, T] \times \mathbb{R}$.

With these facts in hand, we are ready to pass to the limit in (3.8) and conclude that u satisfies the kinetic formulation in the sense of distributions. Note, that (3.51) remains valid also for u so (3.6) follows and, according to the embedding $L^p(\mathbb{T}^N) \hookrightarrow L^1(\mathbb{T}^N)$, for all $p \in [1, \infty)$, we deduce

$$u \in L^p(\Omega \times [0, T], \mathcal{P}, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t; L^p(\mathbb{T}^N)).$$

The proof of Theorem 3.2.10 is complete.

3.A Densely defined martingales

In this section, we present an auxiliary result which is used in the proof of existence of a martingale kinetic solution in Theorem 3.4.12. To be more precise, it solves the following problem: it is needed to show equality of a certain martingale M and a stochastic integral $\int_0^t \sigma \mathrm{d}W$ but the process M is only defined on a dense subset $\mathcal{D}$ of $[0,T]$ containing zero and no continuity property is a priori known. Therefore, one cannot just prove that the quadratic variation of their difference vanishes as it is not well defined.

To begin with, let us fix some notation. Let H, U be separable Hilbert spaces with orthonormal bases $(g_j)_{j\geq 1}$ and $(f_k)_{k\geq 1}$, respectively, and inner products $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_U$, respectively. For simplicity, we will work on a finite-time interval $[0,T]$, $T\in\mathcal{D}$.

Proposition 3.A.1. *Assume that $W(t)=\sum_{k\geq 1}\beta_k(t)f_k$ is a cylindrical Wiener process in U defined on a stochastic basis $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},\mathbb{P})$ with a complete, right-continous filtration. If $(M(t);t\in\mathcal{D})$ is an integrable $(\mathscr{F}_t)$-adapted H-valued stochastic process such that, for any s, $t\in\mathcal{D}$, $s\leq t$, j, $k\geq 1$, $\mathbb{P}$-a.s.,*

$$\begin{aligned}\mathbb{E}\big[\langle M(t)-M(s),g_j\rangle\big|\mathscr{F}_s\big]&=0,\\ \mathbb{E}\bigg[\langle M(t),g_j\rangle^2-\langle M(s),g_j\rangle^2-\int_s^t\|\sigma^* g_j\|_U^2\,\mathrm{d}r\bigg|\mathscr{F}_s\bigg]&=0,\\ \mathbb{E}\bigg[\beta_k(t)\langle M(t),g_j\rangle-\beta_k(s)\langle M(s),g_j\rangle-\int_s^t\langle f_k,\sigma^* g_j\rangle_U\,\mathrm{d}r\bigg|\mathscr{F}_s\bigg]&=0,\end{aligned}\tag{3.52}$$

where σ is an $(\mathscr{F}_t)$-progressively measurable $L_2(U;H)$-valued stochastically integrable process, i.e.

$$\mathbb{E}\int_0^T\|\sigma\|^2_{L_2(U;H)}\,\mathrm{d}r<\infty,\tag{3.53}$$

then

$$M(t)=\int_0^t\sigma\,\mathrm{d}W,\qquad\forall t\in\mathcal{D},\ \mathbb{P}\text{-a.s.}.$$

In particular, M can be defined for all $t\in[0,T]$ such that it has a modification which is a continuous $(\mathscr{F}_t)$-martingale.

Proof. The crucial point to be shown here is the following: for any $(\mathscr{F}_t)$-progressively measurable $L_2(U;H)$-valued process ϕ satisfying (3.53) and any s, $t\in\mathcal{D}$, $s\leq t$, $j\geq 1$, it holds, $\mathbb{P}$-a.s.,

$$\mathbb{E}\bigg[\langle M(t)-M(s),g_j\rangle\Big\langle\int_s^t\phi\,\mathrm{d}W,g_j\Big\rangle-\int_s^t\langle\sigma^* g_j,\phi^* g_j\rangle_U\,\mathrm{d}r\bigg|\mathscr{F}_s\bigg]=0.\tag{3.54}$$

We consider simple processes first. Let ϕ be an $(\mathscr{F}_t)$-adapted simple process with values in finite-dimensional operators of $L(U;H)$ that satisfies (3.53), i.e.

$$\phi(t)=\phi_0\mathbf{1}_{\{0\}}(t)+\sum_{i=0}^I\phi_i\mathbf{1}_{(t_i,t_{i+1}]}(t),\qquad t\in[0,T],$$

where $\{0 = t_0 < t_1 < \cdots < t_I = T\}$ is a division of $[0,T]$ such that $t_i \in \mathcal{D}$, $i = 0,\dots,I$. Then the stochastic integral in (3.54) is given by

$$\begin{aligned}\int_s^t \phi \,\mathrm{d}W &= \phi_{m-1}\big(W(t_m) - W(s)\big)\\ &\qquad + \sum_{i=m}^{n-1} \phi_i\big(W(t_{i+1}) - W(t_i)\big) + \phi_n\big(W(t) - W(t_n)\big)\\ &= \sum_{k\ge 1}\bigg(\phi^k_{m-1}\big(\beta_k(t_m) - \beta_k(s)\big)\\ &\qquad + \sum_{i=m}^{n-1}\phi_i^k\big(\beta_k(t_{i+1}) - \beta_k(t_i)\big) + \phi_n^k\big(\beta_k(t) - \beta_k(t_n)\big)\bigg)\end{aligned}$$

provided $t_{m-1} \le s < t_m$, $t_n \le t < t_{n+1}$, $\phi_i^k = \phi_i f_k$. Next, we write

$$M(t) - M(s) = \big(M(t_m) - M(s)\big) + \sum_{i=m}^{n-1}\big(M(t_{i+1}) - M(t_i)\big) + \big(M(t) - M(t_n)\big)$$

and conclude

$$\begin{aligned}&\mathbb{E}\bigg[\big\langle M(t) - M(s), g_j\big\rangle\Big\langle \int_s^t \phi\,\mathrm{d}W, g_j\Big\rangle\bigg|\mathscr{F}_s\bigg]\\ &\quad= \mathbb{E}\bigg[\big\langle\phi_{m-1}\big(W(t_m) - W(s)\big), g_j\big\rangle\big\langle M(t_m) - M(s), g_j\big\rangle\\ &\qquad + \sum_{i=m}^{n-1}\big\langle\phi_i\big(W(t_{i+1}) - W(t_i)\big), g_j\big\rangle\big\langle M(t_{i+1}) - M(t_i), g_j\big\rangle\\ &\qquad + \big\langle\phi_n\big(W(t) - W(t_n)\big), g_j\big\rangle\big\langle M(t) - M(t_n), g_j\big\rangle\bigg|\mathscr{F}_s\bigg]\end{aligned}\tag{3.55}$$

as one can neglect all the mixed terms due to the martingale property of β_k, $k \ge 1$, and (3.52). Indeed, let $i \in \{m,\dots,n-1\}$ then

$$\begin{aligned}&\mathbb{E}\Big[\big\langle\phi_i(W(t_{i+1}) - W(t_i)), g_j\big\rangle\big\langle M(t_m) - M(s), g_j\big\rangle\Big|\mathscr{F}_s\Big]\\ &\quad= \mathbb{E}\bigg[\mathbb{E}\bigg[\sum_{k\ge 1}\big\langle\phi_i^k\big(\beta_k(t_{i+1}) - \beta_k(t_i)\big), g_j\big\rangle\big\langle M(t_m) - M(s), g_j\big\rangle\bigg|\mathscr{F}_{t_i}\bigg]\bigg|\mathscr{F}_s\bigg]\\ &\quad= \mathbb{E}\bigg[\big\langle M(t_m) - M(s), g_j\big\rangle\sum_{k\ge 1}\langle\phi_i^k, g_j\rangle\,\mathbb{E}\Big[\beta_k(t_{i+1}) - \beta_k(t_i)\Big|\mathscr{F}_{t_i}\Big]\bigg|\mathscr{F}_s\bigg] = 0,\end{aligned}$$

where the interchange of summation with scalar product and expectation, respectively, is justified by the fact that

$$\sum_{k\ge 1}\phi_i^k\big(\beta_k(t_{i+1}) - \beta_k(t_i)\big) = \int_{t_i}^{t_{i+1}}\phi_i\,\mathrm{d}W$$

is convergent in $L^2(\Omega; H)$.

As the next step, we proceed with (3.55). If $i \in \{m, \dots, n-1\}$ then we obtain using again the martingale property of β_k, $k \geq 1$, and (3.52)

$$
\begin{aligned}
&\mathbb{E}\Big[\big\langle \phi_i\big(W(t_{i+1}) - W(t_i)\big), g_j\big\rangle \big\langle M(t_{i+1}) - M(t_i), g_j\big\rangle \Big| \mathscr{F}_s\Big] \\
&\quad = \mathbb{E}\bigg[\mathbb{E}\bigg[\sum_{k\geq 1} \langle \phi_i^k, g_j\rangle \big(\beta_k(t_{i+1}) - \beta_k(t_i)\big)\big\langle M(t_{i+1}) - M(t_i), g_j\big\rangle \bigg| \mathscr{F}_{t_i}\bigg]\bigg| \mathscr{F}_s\bigg] \\
&\quad = \mathbb{E}\bigg[\sum_{k\geq 1} \langle \phi_i^k, g_j\rangle\, \mathbb{E}\Big[\beta_k(t_{i+1})\big\langle M(t_{i+1}), g_j\big\rangle - \beta_k(t_i)\big\langle M(t_i), g_j\big\rangle \Big| \mathscr{F}_{t_i}\Big]\bigg| \mathscr{F}_s\bigg] \\
&\quad = \mathbb{E}\bigg[\sum_{k\geq 1} \langle \phi_i^k, g_j\rangle \int_{t_i}^{t_{i+1}} \big\langle f_k, \sigma^* g_j\big\rangle_U \,\mathrm{d}r \bigg| \mathscr{F}_s\bigg] \\
&\quad = \mathbb{E}\bigg[\sum_{k\geq 1} \langle f_k, \phi_i^* g_j\rangle_U \int_{t_i}^{t_{i+1}} \big\langle f_k, \sigma^* g_j\big\rangle_U \,\mathrm{d}r \bigg| \mathscr{F}_s\bigg] \\
&\quad = \mathbb{E}\bigg[\int_{t_i}^{t_{i+1}} \langle \sigma^* g_j, \phi^* g_j\rangle_U \,\mathrm{d}r \bigg| \mathscr{F}_s\bigg].
\end{aligned}
$$

The remaining terms being dealt with similarly. As a consequence, we see that (3.54) holds true for simple processes and the general case follows by classical arguments using approximation.

Now, we have all in hand to complete the proof. Let $t \in \mathcal{D}$ and set $s = 0$ in (3.52), (3.54), then

$$
\begin{aligned}
\mathbb{E}\bigg(\big\langle M(t), g_j\big\rangle - \Big\langle \int_0^t \sigma \,\mathrm{d}W, g_j\Big\rangle\bigg)^2 &= \mathbb{E}\,\big\langle M(t), g_j\big\rangle^2 \\
- 2\,\mathbb{E}\,\big\langle M(t), g_j\big\rangle \Big\langle \int_0^t \sigma \,\mathrm{d}W, g_j\Big\rangle &+ \mathbb{E}\,\Big\langle \int_0^t \sigma \,\mathrm{d}W, g_j\Big\rangle^2 = 0, \qquad j \geq 1,
\end{aligned}
$$

and the claim follows. □

Chapter 4

A Bhatnagar-Gross-Krook Approximation to Stochastic Scalar Conservation Laws

Abstract: We study a BGK-like approximation to hyperbolic conservation laws forced by a multiplicative white noise. First, we make use of the stochastic characteristics method and establish the existence of a solution for any fixed parameter ε. In the next step, we investigate the limit as ε tends to 0 and show the convergence to the kinetic solution of the limit problem.

Results of this chapter are available as a preprint:

- M. Hofmanová, *A Bhatnagar-Gross-Krook Approximation to Stochastic Scalar Conservation Laws.*

4.1 Introduction

In the present paper, we consider a scalar conservation law with stochastic forcing

$$\begin{aligned} \mathrm{d}u + \operatorname{div}\big(A(u)\big)\mathrm{d}t &= \varPhi(u)\,\mathrm{d}W, \qquad t\in(0,T),\, x\in\mathbb{T}^N,\\ u(0)&=u_0 \end{aligned} \tag{4.1}$$

and study its approximation in the sense of Bhatnagar-Gross-Krook (a BGK-like approximation for short). In particular, we aim to describe the conservation law (4.1) as the hydrodynamic limit of the stochastic BGK model, as the microscopic scale ε goes to 0.

The literature devoted to the deterministic counterpart, i.e. corresponding to the situation $\varPhi=0$, is quite extensive (see [7], [38], [55], [56], [58], [59], [65], [64]). In that case, the BGK model is given as follows

$$\big(\partial_t + a(\xi)\cdot\nabla\big)f^\varepsilon = \frac{\chi_{u^\varepsilon}-f^\varepsilon}{\varepsilon}, \qquad t>0,\, x\in\mathbb{T}^N,\, \xi\in\mathbb{R}, \tag{4.2}$$

where χ_{u^ε}, the so-called equilibrium function, is defined by

$$\chi_{u^\varepsilon}(\xi) = \mathbf{1}_{0<\xi<u^\varepsilon} - \mathbf{1}_{u^\varepsilon<\xi<0},$$

and a is the derivative of A. The differential operator ∇ is with respect to the space variable x. The additional real-valued variable ξ is called velocity; the solution f^ε is then a microscopic density of particles at (t,x) with velocity ξ. The local density of particles is defined by

$$u^\varepsilon(t,x) = \int_{\mathbb{R}} f^\varepsilon(t,x,\xi)\,\mathrm{d}\xi.$$

The collisions of particles are given by the nonlinear kernel on the right hand side of (4.2). The idea is that, as $\varepsilon\to0$, the solutions f^ε of (4.2) converge to χ_u where u is the unique kinetic or entropy solution of the deterministic scalar conservation law.

The addition of the stochastic term to the basic governing equation is rather natural for both practical and theoretical applications. Such a term can be used for instance to account for numerical and empirical uncertainties and therefore stochastic conservation laws has been recently of growing interest, see [6], [16], [20], [36], [44], [70], [76], [78]. The first complete well-posedness result for multi-dimensional scalar conservation laws driven by a general multiplicative noise was obtained by Debussche and Vovelle [16] for the case of kinetic solutions. In the present paper, we extend this result and show that the kinetic solution is the macroscopic limit of stochastic BGK approximations. As the latter are much simpler equations that can be solved explicitly, this analysis can be used for developing innovative numerical schemes for hyperbolic conservation laws.

The BGK model in the stochastic case reads

$$\begin{aligned} \mathrm{d}F^\varepsilon + a(\xi)\cdot\nabla F^\varepsilon\,\mathrm{d}t &= \frac{\mathbf{1}_{u^\varepsilon>\xi}-F^\varepsilon}{\varepsilon}\,\mathrm{d}t - \partial_\xi F^\varepsilon\varPhi\,\mathrm{d}W - \frac12\partial_\xi\big(G^2(-\partial_\xi F^\varepsilon)\big)\,\mathrm{d}t,\\ F^\varepsilon(0)&=F^\varepsilon_0, \end{aligned} \tag{4.3}$$

where the function F^ε corresponds to $f^\varepsilon+\mathbf{1}_{0>\xi}$, the local density u^ε is given as above, and the function G^2 will be defined in (4.4). Note, that setting $\varPhi=0$ in (4.3) yields an equation which is equivalent to the deterministic BGK model (4.2). Our purpose here is twofold. First, we make use of the stochastic characteristics method as developed by

Kunita in [50] to study a certain auxiliary problem. With this in hand, we fix ε and prove the existence of a unique weak solution to the stochastic BGK model (4.3). Second, we establish a series of estimates uniform in ε which together with the results of Debussche and Vovelle [16] justify the limit argument, as $\varepsilon \to 0$, and give the convergence of the weak solutions of (4.3) to the kinetic solution of (4.1).

Let us make some comments on the deterministic BGK model (4.2). Even though the general concept of the proof is analogous, we point out that the techniques required by the stochastic case are significantly different. In particular, the characteristic system for the deterministic BGK model consists of independent equations

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = a_i(\xi), \qquad i = 1, \dots, N,$$

and the ξ-coordinate of the characteristic curve is constant. Accordingly, it is much easier to control the behavior of f^ε for large ξ. Namely, if the initial data f_0^ε are compactly supported (in ξ), the same remains valid also for the solution itself and also the convergence proof simplifies. On the contrary, in the stochastic case, the ξ-coordinate of the characteristic curve is governed by an SDE and therefore this property is, in general, lost. Similar issues has to be dealt with in order to obtain all the necessary uniform estimates. To overcome this difficulty, it was needed to develop a suitable method to control the decay at infinity in connection with the remaining variables ω, t, x. (cf. Proposition 4.5.3).

There is another difficulty coming from the complex structure of the characteristic system for the stochastic BGK model (4.3). Namely, the finite speed of propagation that is an easy consequence of boundedness of the solution u of the conservation law in the deterministic case (see for instance [65]) is no longer valid and therefore some growth assumptions on the transport coefficient a are in place. The hypothesis of bounded derivatives is natural for the stochastic characteristics method as it implies the existence of global stochastic flows. Even though this already includes one important example of Burgers' equation it is of essential interest to handle also more general coefficients having polynomial growth. This was achieved by a suitable cut-off procedure which also guarantees all the necessary estimates.

The exposition is organized as follows. In Section 4.2, we introduce the basic setting and state the main result, Theorem 4.2.1. In order to make the paper more self-contained, Section 4.3 provides a brief overview of two concepts which are the keystones of our proof of existence and convergence of the BGK model. On the one hand, it is the notion of kinetic solution to stochastic hyperbolic conservation laws, on the other hand, the method of stochastic characteristics for first-order linear SPDEs. Section 4.4 is mainly devoted to the existence proof for stochastic BGK model, however, in the Subsection 4.4.2 we establish some important estimates useful in Section 4.5. This final section contains technical details of the passage to the limit and completes the proof of Theorem 4.2.1.

4.2 Setting and the main result

We now give the precise assumptions on each of the terms appearing in the above equations (4.1) and (4.3). We work on a finite-time interval $[0,T]$, $T>0$, and consider periodic boundary conditions: $x \in \mathbb{T}^N$ where $\mathbb{T}^N$ is the N-dimensional torus. The flux function

$$A = (A_1, \dots, A_N) : \mathbb{R} \longrightarrow \mathbb{R}^N$$

is supposed to be of class $C^{4,\eta}$, for some $\eta>0$, with a polynomial growth of its first derivative, denoted by $a=(a_1,\dots,a_N)$.

Regarding the stochastic term, let $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},\mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration. The initial datum may be random in general, i.e. $\mathscr{F}_0$-measurable, and we assume $u_0\in L^p(\Omega;L^p(\mathbb{T}^N))$ for all $p\in[1,\infty)$. As we intend to apply the stochastic characteristics method developed by Kunita [50], we restrict ourselves to finite-dimensional noise. However, our results extend to infinite-dimensional setting once the corresponding properties of stochastic flows are established. Let $\mathfrak{U}$ be a finite-dimensional Hilbert space and $(e_k)_{k=1}^d$ its orthonormal basis. The process W is a d-dimensional $(\mathscr{F}_t)$-Wiener process: $W(t)=\sum_{k=1}^d\beta_k(t)\,e_k$ with $(\beta_k)_{k=1}^d$ being mutually independent real-valued standard Wiener processes relative to $(\mathscr{F}_t)_{t\geq 0}$. The diffusion coefficient $\varPhi$ is then defined as

$$\begin{aligned}\varPhi(z):\mathfrak{U}&\longrightarrow L^2(\mathbb{T}^N)\\ h&\longmapsto\sum_{k=1}^d g_k(\cdot,z(\cdot))\langle e_k,h\rangle,\qquad z\in L^2(\mathbb{T}^N),\end{aligned}$$

where the functions $g_1,\dots,g_d:\mathbb{T}^N\times\mathbb{R}\to\mathbb{R}$ are of class $C^{4,\eta}$, for some $\eta>0$, with linear growth and bounded derivatives of all orders. Under these assumptions, the following estimate holds true

$$G^2(x,\xi)=\sum_{k=1}^d|g_k(x,\xi)|^2\leq C\big(1+|\xi|^2\big),\qquad x\in\mathbb{T}^N,\ \xi\in\mathbb{R}.\tag{4.4}$$

However, in order to get all the necessary estimates (cf. Corollary 4.4.11, Remark 4.4.12), we restrict ourselves to two special cases: either

$$g_k(x,0)=0,\qquad x\in\mathbb{T}^N,\ k=1,\dots,d,\tag{4.5}$$

hence (4.4) rewrites as

$$G^2(x,\xi)\leq C|\xi|^2,\qquad x\in\mathbb{T}^N,\ \xi\in\mathbb{R},$$

or we strengthen (4.4) in the following way

$$G^2(x,\xi)\leq C,\qquad x\in\mathbb{T}^N,\ \xi\in\mathbb{R}.\tag{4.6}$$

Note, that the latter is satisfied for instance in the case of additive noise.

In this setting, we can assume without loss of generality that the σ-algebra $\mathscr{F}$ is countably generated and $(\mathscr{F}_t)_{t\geq 0}$ is the completed filtration generated by the Wiener process and the initial condition. Let us denote by $\mathcal{P}$ the predictable σ-algebra on $\Omega\times[0,T]$ associated to $(\mathscr{F}_t)_{t\geq 0}$ and by $\mathcal{P}_s$ the predictable σ-algebra on $\Omega\times[s,T]$ associated to $(\mathscr{F}_t)_{t\geq s}$. For notational simplicity, we write $L^\infty_{\mathcal{P}_s}(\Omega\times[s,T]\times\mathbb{T}^N\times\mathbb{R})$ to denote[1]

$$L^\infty\big(\Omega\times[s,T]\times\mathbb{T}^N\times\mathbb{R},\mathcal{P}_s\otimes\mathcal{B}(\mathbb{T}^N)\otimes\mathcal{B}(\mathbb{R}),\mathrm{d}\mathbb{P}\otimes\mathrm{d}t\otimes\mathrm{d}x\otimes\mathrm{d}\xi\big).$$

Concerning the initial data for the BGK model (4.3), one possibility is to consider simply $F_0^\varepsilon=\mathbf{1}_{u_0>\xi}$, however, one can also take some suitable approximations of $\mathbf{1}_{u_0>\xi}$. Namely, let $\{u_0^\varepsilon;\,\varepsilon\in(0,1)\}$ be a set of approximate $\mathscr{F}_0$-measurable initial data, which is bounded in $L^p(\Omega;L^p(\mathbb{T}^N))$ for all $p\in[1,\infty)$, and assume in addition that $u_0^\varepsilon\to u_0$ in

[1] $\mathcal{B}(\mathbb{T}^N)$ and $\mathcal{B}(\mathbb{R})$, respectively, denotes the Borel σ-algebra on $\mathbb{T}^N$ and $\mathbb{R}$, respectively.

$L^1(\Omega; L^1(\mathbb{T}^N))$. Thus, setting $F_0^\varepsilon = \mathbf{1}_{u_0^\varepsilon>\xi}$, $f_0^\varepsilon = \chi_{u_0^\varepsilon}$ yields the convergence $f_0^\varepsilon \to f_0 = \chi_{u_0}$ in $L^1(\Omega \times \mathbb{T}^N \times \mathbb{R})$.

Let us close this section by stating the main result to be proved precisely.

Theorem 4.2.1 (Hydrodynamic limit of the stochastic BGK model)**.** *Let the above assumptions hold true. Then, for any $\varepsilon > 0$, there exists $F^\varepsilon \in L^\infty_{\mathcal{P}}(\Omega \times [0,T] \times \mathbb{T}^N \times \mathbb{R})$ which is a unique weak solution to the stochastic BGK model (4.3) with initial condition $F_0^\varepsilon = \mathbf{1}_{u_0^\varepsilon>\xi}$. Furthermore, if $f^\varepsilon = F^\varepsilon - \mathbf{1}_{0>\xi}$ then (f^ε) converges in $L^p(\Omega \times [0,T] \times \mathbb{T}^N \times \mathbb{R})$, for all $p \in [1,\infty)$, to the equilibrium function χ_u, where u is the unique kinetic solution to the stochastic hyperbolic conservation law (4.1). Besides, the local densities (u^ε) converge to the kinetic solution u in $L^p(\Omega \times [0,T] \times \mathbb{T}^N)$, for all $p \in [1,\infty)$.*

Throughout the paper, we use the letter C to denote a generic positive constant, which can depend on different quantities but ε and may change from one line to another. We also employ a shortened notation for various L^p-type norms, e.g. we write $\|\cdot\|_{L^p_{\omega,x,\xi}}$ for the norm in $L^p(\Omega \times \mathbb{T}^N \times \mathbb{R})$ and similarly for other spaces.

4.3 Preliminary results

As we are going to apply the well-posedness theory for kinetic solutions of hyperbolic scalar conservation laws (4.1) as well as the theory of stochastic flows generated by stochastic differential equations, we provide a brief overview of these two concepts.

4.3.1 Kinetic formulation for scalar conservation laws

The main reference for this subsection is the paper of Debussche and Vovelle [16]. For further reading about the kinetic approach used in different settings, we refer the reader to [13], [32], [55], [56], or [64]. In the paper [16], the notion of kinetic and generalized kinetic solution to (4.1) was introduced and the existence, uniqueness and continuous dependence on initial data were proved. In the following, we present the main ideas and results while skipping all the technicalities.

Let u be a smooth solution to (4.1). It follows from the Itô formula that u also satisfies the kinetic formulation of (4.1)

$$\partial_t F + a(\xi)\cdot \nabla F = \delta_{u=\xi}\Phi(u)\dot{W} + \partial_\xi\left(m - \frac{1}{2}G^2\delta_{u=\xi}\right), \tag{4.7}$$

where $F = \mathbf{1}_{u>\xi}$ and m is an unknown kinetic measure, i.e. a random nonnegative bounded Borel measure on $[0,T] \times \mathbb{T}^N \times \mathbb{R}$ that vanishes for large ξ in the following sense: if $B_R^c = \{\xi \in \mathbb{R}; |\xi| \geq R\}$ then

$$\lim_{R\to\infty} \mathbb{E}\, m\big(\mathbb{T}^N \times [0,T] \times B_R^c\big) = 0.$$

Hence we arrive at the notion of kinetic solution: $u \in L^p(\Omega \times [0,T], \mathcal{P}, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t; L^p(\mathbb{T}^N))$ is said to be a kinetic solution to (4.1) provided $F = \mathbf{1}_{u>\xi}$ is a solution, in the sense of distributions over $[0,T] \times \mathbb{T}^N \times \mathbb{R}$, to the kinetic formulation (4.7) for some kinetic measure m. Replacing the indicator function by a general kinetic function F we obtain the definition of a generalized kinetic solution. It corresponds to the situation where one does not know the exact value of $u(t,x)$ but only its law given by a probability measure $\nu_{t,x}$. More precisely, let $F(t)$, $t \in [0,T]$, be a kinetic function on $\Omega \times \mathbb{T}^N \times \mathbb{R}$ and $\nu_{t,x}(\xi) = -\partial_\xi F(t,x,\xi)$. Then F is a generalized kinetic solution to (4.1) provided:

$F(0)=\mathbf{1}_{u_0>\xi}$ and for any test function $\varphi\in C_c^\infty([0,T)\times\mathbb{T}^N\times\mathbb{R})$,

$$\begin{aligned}\int_0^T\big\langle F(t),\partial_t\varphi(t)\big\rangle\,\mathrm{d}t+\big\langle F(0),\varphi(0)\big\rangle+\int_0^T\big\langle F(t),a(\xi)\cdot\nabla\varphi(t)\big\rangle\,\mathrm{d}t\\ =-\sum_{k=1}^d\int_0^T\int_{\mathbb{T}^N}\int_{\mathbb{R}}g_k(x,\xi)\varphi(t,x,\xi)\,\mathrm{d}\nu_{t,x}(\xi)\,\mathrm{d}x\,\mathrm{d}\beta_k(t)\\ -\frac{1}{2}\int_0^T\int_{\mathbb{T}^N}\int_{\mathbb{R}}G^2(x,\xi)\partial_\xi\varphi(t,x,\xi)\mathrm{d}\nu_{t,x}(\xi)\,\mathrm{d}x\,\mathrm{d}t+m(\partial_\xi\varphi)\end{aligned}\tag{4.8}$$

holds true $\mathbb{P}$-a.s.. The assumptions considered in [16] are the following: the flux function A is of class C^1 with a polynomial growth of its derivative; the process W is a (generally infinite-dimensional) cylindrical Wiener process, i.e. $W(t)=\sum_{k\geq1}\beta_k(t)e_k$ with $(\beta_k)_{k\geq1}$ being mutually independent real-valued standard Wiener processes and $(e_k)_{k\geq1}$ a complete orthonormal system in a separable Hilbert space $\mathfrak{U}$; the mapping $\Phi(z):\mathfrak{U}\to L^2(\mathbb{T}^N)$ is defined for each $z\in L^2(\mathbb{T}^N)$ by $\Phi(z)e_k=g_k(\cdot,z(\cdot))$ where $g_k\in C(\mathbb{T}^N\times\mathbb{R})$ and the following conditions

$$\sum_{k\geq1}|g_k(x,\xi)|^2\leq C\big(1+|\xi|^2\big),$$

$$\sum_{k\geq1}|g_k(x,\xi)-g_k(y,\zeta)|^2\leq C\big(|x-y|^2+|\xi-\zeta|h(|\xi-\zeta|)\big),$$

are fulfilled for every $x,y\in\mathbb{T}^N$, $\xi,\zeta\in\mathbb{R}$, with h being a continuous nondecreasing function on $\mathbb{R}_+$ satisfying, for some $\alpha>0$,

$$h(\delta)\leq C\delta^\alpha,\quad \delta<1.$$

Under these hypotheses, the well-posedness result [16, Theorem 11, Theorem 19] states: For any $u_0\in L^p(\Omega\times\mathbb{T}^N)$ for all $p\in[1,\infty)$ there exists a unique kinetic solution to (4.1). Besides, any generalized kinetic solution F is actually a kinetic solution, i.e. there exists a process u such that $F=\mathbf{1}_{u>\xi}$. Moreover, if u_1, u_2 are kinetic solutions with initial data $u_{1,0}$ and $u_{2,0}$, respectively, then for all $t\in[0,T]$

$$\mathbb{E}\|u_1(t)-u_2(t)\|_{L^1_x}\leq\mathbb{E}\|u_{1,0}-u_{2,0}\|_{L^1_x}.$$

4.3.2 Stochastic flows and stochastic characteristics method

The results mentioned in this subsection are due to Kunita and can be found in [49] and [50]. To begin with, we introduce some notation. We denote by $C_b^{l,\delta}(\mathbb{R}^d)$ the space of all l-times continuously differentiable functions with bounded derivatives up to order l (the function itself is only required to be of linear growth) and δ-Hölder continuous l-th derivatives.

Let $B_t=(B_t^1,\dots,B_t^m)$ be an m-dimensional Wiener process and let $b^k:\mathbb{R}^d\to\mathbb{R}^d$, $k=0,\dots,m$. We study the following system of Stratonovich's stochastic differential equations

$$\mathrm{d}\phi_t=b^0(\phi_t)\,\mathrm{d}t+\sum_{k=1}^m b^k(\phi_t)\circ\mathrm{d}B_t^k.\tag{4.9}$$

Under the hypothesis that $b^1,\dots,b^m\in C_b^{l+1,\delta}(\mathbb{R}^d)$ and $b^0\in C_b^{l,\delta}(\mathbb{R}^d)$ for some $l\geq1$ and $\delta>0$, and for any given $y\in\mathbb{R}^d$, $s\in[0,T]$, the problem (4.9) possesses a unique solution starting from y at time s. Let us denote this solution by $\phi_{s,t}(y)$. It enjoys several important properties. Namely, it is a continuous $C^{l,\varepsilon}$-semimartingale for any $\varepsilon < \delta$ and defines a forward Brownian stochastic flow of C^l-diffeomorphisms, i.e. there exists a null set N of Ω such that for any $\omega \in N^c$, the family of continuous maps $\{\phi_{s,t}(\omega);\, 0 \leq s \leq t \leq T\}$ satisfies

(i) $\phi_{s,t}(\omega) = \phi_{r,t}(\omega) \circ \phi_{s,r}(\omega)$ for all $0 \leq s \leq r \leq t \leq T$,

(ii) $\phi_{s,s}(\omega) = \mathrm{Id}$ for all $0 \leq s \leq T$,

(iii) $\phi_{s,t}(\omega) : \mathbb{R}^d \to \mathbb{R}^d$ is l-times differentiable with respect to y, for all $0 \leq s \leq t \leq T$, and the derivatives are continuous in (s, t, y),

(iv) $\phi_{s,t}(\omega) : \mathbb{R}^d \to \mathbb{R}^d$ is a C^l-diffeomorphism for all $0 \leq s \leq t \leq T$,

(v) $\phi_{t_i,t_{i+1}}$, $i = 0, \dots, n-1$, are independent random variables for any $0 \leq t_0 \leq \cdots \leq t_n \leq T$.

Therefore, for each $0 \leq s \leq t \leq T$, the mapping $\phi_{s,t}(\omega)$ has the inverse $\rho_{s,t}(\omega) = \phi_{s,t}(\omega)^{-1}$ which satisfies

(vi) $\rho_{s,t}(\omega) : \mathbb{R}^d \to \mathbb{R}^d$ is l-times differentiable with respect to y, for all $0 \leq s \leq t \leq T$, and the derivatives are continuous in (s, t, y),

(vii) $\rho_{s,t}(\omega) = \rho_{s,r}(\omega) \circ \rho_{r,t}(\omega)$ for all $0 \leq s \leq r \leq t \leq T$,

and consequently $\rho_{s,t}$ is a stochastic flow of C^l-diffeomorphisms for the backward direction. Indeed, the following holds true: For any $0 \leq s \leq t \leq T$, the process $\rho_{s,t}(y)$ satisfies the backward Stratonovich stochastic differential equation with the terminal condition y

$$\rho_{s,t}(y) = y - \int_s^t b^0\big(\rho_{r,t}(y)\big)\,\mathrm{d}r - \sum_{k=1}^m \int_s^t b^k\big(\rho_{r,t}(y)\big) \circ \hat{\mathrm{d}}B_r^k,$$

where the last term is a backward Stratonovich integral defined by Kunita [50] using the time-reversing method. To be more precise, the Brownian motion B is regarded as a backward martingale with respect to its natural two parametric filtration

$$\sigma\big(B_{r_1} - B_{r_2};\, s \leq r_1, r_2 \leq t\big), \qquad 0 \leq s \leq t \leq T,$$

the integral is then defined similarly to the forward case and both stochastic flows $\phi_{s,t}$ as well as $\rho_{s,t}$ are adapted to this filtration. Furthermore, we have a growth control for both forward and backward stochastic flow. Fix arbitrary $\delta \in (0, 1)$, then the following convergences hold uniformly in s, t, $\mathbb{P}$-a.s.,

$$\lim_{|y|\to\infty} \frac{|\phi_{s,t}(y)|}{(1 + |y|)^{1+\delta}} = 0, \qquad \lim_{|y|\to\infty} \frac{|\rho_{s,t}(y)|}{(1 + |y|)^{1+\delta}} = 0,$$

$$\lim_{|y|\to\infty} \frac{(1 + |y|)^{\delta}}{1 + |\phi_{s,t}(y)|} = 0, \qquad \lim_{|y|\to\infty} \frac{(1 + |y|)^{\delta}}{1 + |\rho_{s,t}(y)|} = 0.$$

In the remainder of this subsection we will discuss the stochastic characteristics method where the theory of stochastic flows plays an important role. We restrict our attention to a first-order linear stochastic partial differential equation of the form

$$\begin{aligned} \mathrm{d}v &= b^0(y)\cdot\nabla_y v\,\mathrm{d}t + \sum_{k=1}^m b^k(y)\cdot\nabla_y v\circ\mathrm{d}B_t^k, \\ v(0) &= v_0, \end{aligned} \tag{4.10}$$

with coefficients $b^k:\mathbb{R}^d\to\mathbb{R}^d$, $k=0,\dots,m$. The associated stochastic characteristic system is defined by a system of Stratonovich stochastic differential equations

$$\mathrm{d}\phi_t = b^0(\phi_t)\,\mathrm{d}t + \sum_{k=1}^m b^k(\phi_t)\circ\mathrm{d}B_t^k, \tag{4.11}$$

A solution of (4.11) starting at y is the so-called stochastic characteristic curve of (4.10) and will be denoted by $\phi_t(y)$. Assume that $b^1,\dots,b^m\in C_b^{l+1,\delta}(\mathbb{R}^d)$ and $b^0\in C_b^{l,\delta}(\mathbb{R}^d)$ for some $l\geq 3$ and $\delta>0$. If the initial function v_0 lies in $C^{l,\delta}(\mathbb{R}^d)$, then the problem (4.10) has a unique strong solution which is a continuous $C^{l,\varepsilon}$-semimartingale for some $\varepsilon>0$ and is represented by

$$v(t,y) = v_0\big(\phi_t^{-1}(y)\big),\qquad t\in[0,T], \tag{4.12}$$

where the inverse mapping ϕ_t^{-1} is well defined according to the previous paragraph. It satisfies (4.10) in the following sense

$$v(t,y) = v_0(y) + b^0(y)\cdot\int_0^t \nabla_y v(r,y)\,\mathrm{d}r + \sum_{k=1}^m b^k(y)\cdot\int_0^t\nabla_y v(r,y)\circ\mathrm{d}B_r^k.$$

Moreover, if the initial condition v_0 is rapidly decreasing then so does the solution itself and

$$\mathbb{E}\sup_{t\in[0,T]}\bigg(\int_{\mathbb{R}^d}|v(t,y)|(1+|y|)^n\,\mathrm{d}y\bigg)^p<\infty,\qquad \forall n\in\mathbb{N}_0,\ p\in[1,\infty).$$

The choice of the Stratonovich integral is more natural here and is given by application of the Itô-Wentzell-type formula in the proof of the explicit representation of the solution (4.12). Indeed, in this case it is close to the classical differential rule formula for composite functions (cf. [49, Theorem I.8.1, Theorem I.8.3]).

4.4 Solution to the stochastic BGK model

This section is devoted to the existence proof for the stochastic BGK model (4.3). Let us start with the definition of its solution.

Definition 4.4.1. Let $\varepsilon>0$. Then $F^\varepsilon\in L^\infty_{\mathcal{P}}(\Omega\times[0,T]\times\mathbb{T}^N\times\mathbb{R})$ satisfying $F^\varepsilon-\mathbf{1}_{0>\xi}\in L^1(\Omega\times[0,T]\times\mathbb{T}^N\times\mathbb{R})$ is called a weak solution to the stochastic BGK model (4.3) with initial condition F_0^ε provided the following holds true for a.e. $t\in[0,T]$, $\mathbb{P}$-a.s.,

$$\begin{aligned} \big\langle F^\varepsilon(t),\varphi\big\rangle &= \big\langle F_0^\varepsilon,\varphi\big\rangle + \int_0^t\big\langle F^\varepsilon(s),a\cdot\nabla\varphi\big\rangle\,\mathrm{d}s \\ &+\frac{1}{\varepsilon}\int_0^t\big\langle\mathbf{1}_{u^\varepsilon(t)>\xi}-F^\varepsilon(t),\varphi(t)\big\rangle\,\mathrm{d}t + \sum_{k=1}^d\int_0^t\big\langle F^\varepsilon(s),\partial_\xi(g_k\varphi)\big\rangle\,\mathrm{d}\beta_k(s) \\ &+\frac{1}{2}\int_0^t\big\langle F^\varepsilon(s),\partial_\xi(G^2\partial_\xi\varphi)\big\rangle\,\mathrm{d}s. \end{aligned}$$

Remark 4.4.2. In particular, for any $\varphi\in C_c^\infty(\mathbb{T}^N\times\mathbb{R})$, there exists a representative of $\langle F^\varepsilon(t),\varphi\rangle\in L^\infty(\Omega\times[0,T])$ which is a continuous stochastic process.

In order to solve the stochastic BGK model (4.3), we intend to employ the stochastic characterics method introduced in the previous section. Hence we need to reformulate the problem in Stratonovich form. It will be seen from the following lemma (see Corollary 4.4.4) that on the level of above defined weak solutions the problem (4.3) is equivalent to

$$\begin{aligned}\mathrm{d}F^\varepsilon+a(\xi)\cdot\nabla F^\varepsilon\,\mathrm{d}t&=\frac{\mathbf{1}_{u^\varepsilon>\xi}-F^\varepsilon}{\varepsilon}\,\mathrm{d}t-\partial_\xi F^\varepsilon\varPhi\circ\mathrm{d}W+\frac{1}{4}\partial_\xi F^\varepsilon\partial_\xi G^2\,\mathrm{d}t,\\ F^\varepsilon(0)&=F_0^\varepsilon.\end{aligned}$$

Lemma 4.4.3. *If X be a $C^1(\mathbb{T}^N\times\mathbb{R})$-valued continuous $(\mathscr{F}_t)$-semimartingale whose martingale part is given by $-\int_0^t\partial_\xi X\varPhi\,\mathrm{d}W$, then*

$$-\int_0^t\partial_\xi X\varPhi\,\mathrm{d}W+\frac{1}{2}\int_0^t\partial_\xi\big(G^2\partial_\xi X\big)\mathrm{d}t=-\int_0^t\partial_\xi X\varPhi\circ\mathrm{d}W+\frac{1}{4}\int_0^t\partial_\xi X\partial_\xi G^2\mathrm{d}t.\qquad(4.13)$$

Moreover, the same is valid in the sense of distributions as well: let X be a $\mathcal{D}'(\mathbb{T}^N\times\mathbb{R})$-valued continuous $(\mathscr{F}_t)$-semimartingale whose martingale part is given by

$$-\int_0^t\partial_\xi X\varPhi\,\mathrm{d}W,$$

i.e. $\langle X(t),\varphi\rangle$ is a continuous $(\mathscr{F}_t)$-semimartingale with martingale part

$$-\int_0^t\langle\partial_\xi X\varPhi,\varphi\rangle\,\mathrm{d}W$$

for any $\varphi\in C_c^\infty(\mathbb{T}^N\times\mathbb{R})$. Then (4.13) holds true in $\mathcal{D}'(\mathbb{T}^N\times\mathbb{R})$.

Proof. We will only prove the second part of the statement as the first one is straightforward and follows similar arguments. Let us recall the relation between Itô and Stratonovich integrals (see [49] or [50]). Let Y be a continuous local semimartingale and Ψ be a continuous semimartingale. Then the Stratonovich integral is well defined and satisfies

$$\int_0^t\Psi\circ\mathrm{d}Y=\int_0^t\Psi\,\mathrm{d}Y+\frac{1}{2}\langle\!\langle\Psi,Y\rangle\!\rangle_t,$$

where $\langle\!\langle\cdot,\cdot\rangle\!\rangle_t$ denotes the cross-variation process. Therefore, we need to calculate the cross variation of $-\partial_\xi Xg_k$ and the Wiener process β_k, $k=1,\dots,d$. Towards this end, we take a test function $\varphi\in C_c^\infty(\mathbb{T}^N\times\mathbb{R})$ and derive the martingale part of $\langle\partial_\xi Xg_k,\varphi\rangle$ (in the following, we emphasize only the corresponding martingale parts).

$$\begin{aligned}\langle X,\varphi\rangle&=\cdots-\int_0^t\big\langle\partial_\xi Xg_k,\varphi\big\rangle\,\mathrm{d}\beta_k(s),\\ \langle X,g_k\varphi\rangle&=\cdots-\int_0^t\big\langle\partial_\xi Xg_k,g_k\varphi\big\rangle\,\mathrm{d}\beta_k(s),\\ \langle\partial_\xi X,g_k\varphi\rangle&=\cdots+\int_0^t\big\langle\partial_\xi Xg_k,\partial_\xi(g_k\varphi)\big\rangle\,\mathrm{d}\beta_k(s),\end{aligned}$$

where

$$\begin{aligned}\big\langle\partial_\xi X g_k,\partial_\xi(g_k\varphi)\big\rangle&=-\big\langle\partial_\xi(\partial_\xi X g_k),g_k\varphi\big\rangle\\&=-\big\langle\partial_\xi^2 X g_k^2,\varphi\big\rangle-\frac{1}{2}\big\langle\partial_\xi X\partial_\xi g_k^2,\varphi\big\rangle\\&=-\big\langle\partial_\xi(g_k^2\partial_\xi X),\varphi\big\rangle+\frac{1}{2}\big\langle\partial_\xi X\partial_\xi g_k^2,\varphi\big\rangle.\end{aligned}$$

Consequently

$$\big\langle\!\big\langle\langle-\partial_\xi X g_k,\varphi\rangle,\beta_k\big\rangle\!\big\rangle_t=\int_0^t\big\langle\partial_\xi(g_k^2\partial_\xi X),\varphi\big\rangle\,\mathrm{d}s-\frac{1}{2}\int_0^t\big\langle\partial_\xi X\partial_\xi g_k^2,\varphi\big\rangle\,\mathrm{d}s$$

and the claim follows by summing up over k. □

Corollary 4.4.4. *Let $\varepsilon>0$. If $F^\varepsilon\in L^\infty_{\mathcal{P}}(\Omega\times[0,T]\times\mathbb{T}^N\times\mathbb{R})$ is such that $F^\varepsilon-\mathbf{1}_{0>\xi}\in L^1(\Omega\times[0,T]\times\mathbb{T}^N\times\mathbb{R})$ then it is a weak solution to (4.3) if and only if, for any $\varphi\in C_c^\infty(\mathbb{T}^N\times\mathbb{R})$, there exists a representative of $\langle F^\varepsilon(t),\partial_\xi(g_k\varphi)\rangle\in L^\infty(\Omega\times[0,T])$ which is a continuous $(\mathscr{F}_t)$-semimartingale and the following holds true for a.e. $t\in[0,T]$, $\mathbb{P}$-a.s.,*

$$\begin{aligned}\big\langle F^\varepsilon(t),\varphi\big\rangle&=\big\langle F_0^\varepsilon,\varphi\big\rangle+\int_0^t\big\langle F^\varepsilon(s),a\cdot\nabla\varphi\big\rangle\,\mathrm{d}s\\&+\frac{1}{\varepsilon}\int_0^t\big\langle\mathbf{1}_{u^\varepsilon(t)>\xi}-F^\varepsilon(t),\varphi(t)\big\rangle\,\mathrm{d}t+\sum_{k=1}^d\int_0^t\big\langle F^\varepsilon(s),\partial_\xi(g_k\varphi)\big\rangle\circ\mathrm{d}\beta_k(s)\\&-\frac{1}{4}\int_0^t\big\langle F^\varepsilon(s),\partial_\xi(\varphi\partial_\xi G^2)\big\rangle\,\mathrm{d}s.\end{aligned}$$

As the first step in order to show the existence of a solution to the stochastic BGK model, we shall study the following auxiliary problem:

$$\begin{aligned}\mathrm{d}X+a(\xi)\cdot\nabla X\,\mathrm{d}t&=-\partial_\xi X\varPhi\circ\mathrm{d}W+\frac{1}{4}\partial_\xi X\partial_\xi G^2\,\mathrm{d}t,\\X(s)&=X_0.\end{aligned}\tag{4.14}$$

It will be shown in Corollary 4.4.10 that this problem possesses a unique weak solution provided $X_0\in L^\infty(\Omega\times\mathbb{T}^N\times\mathbb{R})$. Let

$$\mathcal{S}=\{\mathcal{S}(t,s);\,0\leq s\leq t\leq T\}$$

be its solution operator, i.e. for any $0\leq s\leq t\leq T$ we define $\mathcal{S}(t,s)X_0$ to be the solution to (4.14). Then we have the following existence result for the stochastic BGK model.

Theorem 4.4.5. *For any $\varepsilon>0$, there exists a unique weak solution of the stochastic BGK model (4.3) and is represented by*

$$F^\varepsilon(t)=\mathrm{e}^{-\frac{t}{\varepsilon}}\mathcal{S}(t,0)F_0^\varepsilon+\frac{1}{\varepsilon}\int_0^t\mathrm{e}^{-\frac{t-s}{\varepsilon}}\mathcal{S}(t,s)\mathbf{1}_{u^\varepsilon(s)>\xi}\,\mathrm{d}s.\tag{4.15}$$

The proof of Theorem 4.4.5 will be divided into several steps. First, we have to concentrate on the problem (4.14).

4.4.1 Application of the stochastic characteristics method

In this subsection, we prove the existence of a unique solution to (4.14) and study the behavior of the solution operator $\mathcal{S}$. The equation (4.14) is a first-order linear stochastic partial differential equation of the form (4.10), however, the coefficient a, as well as $\partial_\xi G^2$ in the case of (4.5), is not supposed to have bounded derivatives. For this purpose we introduce the following truncated problem: let (k_R) be a smooth truncation on $\mathbb{R}$, i.e. let $k_R(\xi) = k(R^{-1}\xi)$, where k is a smooth function with compact support satisfying $0 \leq k \leq 1$ and

$$k(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq \frac{1}{2}, \\ 0, & \text{if } |\xi| \geq 1, \end{cases}$$

and define $g_k^R(x,\xi) = g_k(x,\xi)k_R(\xi)$, $k = 1,\dots,d$, and $a^R(\xi) = a(\xi)k_R(\xi)$. Coefficients $\varPhi^R$ and $G^{R,2}$, respectively, can be defined similarly as $\varPhi$ and G^2, respectively, using g_k^R instead of g_k.[2] Then

$$\begin{aligned} \mathrm{d}X + a^R(\xi)\cdot\nabla X\,\mathrm{d}t &= -\partial_\xi X\varPhi^R \circ \mathrm{d}W + \frac{1}{4}\partial_\xi X \partial_\xi G^{R,2}\,\mathrm{d}t, \\ X(s) &= X_0 \end{aligned} \tag{4.16}$$

can be solved by the method of stochastic characteristics. Indeed, the stochastic characteristic system associated with (4.16) is defined by the following system of Stratonovich's stochastic differential equations

$$\begin{aligned} \mathrm{d}\varphi_t^0 &= -\frac{1}{4}\partial_\xi G^{R,2}(\varphi_t)\,\mathrm{d}t + \sum_{k=1}^d g_k^R(\varphi_t)\circ\mathrm{d}\beta_k(t), \\ \mathrm{d}\varphi_t^i &= a_i^R(\varphi_t^0)\,\mathrm{d}t, \qquad i = 1,\dots,N, \end{aligned} \tag{4.17}$$

where the processes φ_t^0 and φ_t^i, $i = 1,\dots,N$, respectively, describe the evolution of the ξ-coordinate and x^i-coordinate, $i = 1,\dots,N$, respectively, of the characteristic curve.

Let us denote by $\varphi_{s,t}^R(x,\xi)$ the solution of (4.17) starting from (x,ξ) at time s. Then φ^R defines a stochastic flow of C^3-diffeomorphisms and we denote by ψ^R the corresponding inverse flow. It is the solution to the backward problem

$$\begin{aligned} \mathrm{d}\psi_t^0 &= \frac{1}{4}\partial_\xi G^{R,2}(\psi_t)\,\hat{\mathrm{d}}t - \sum_{k=1}^d g_k^R(\psi_t)\circ\hat{\mathrm{d}}\beta_k(t), \\ \mathrm{d}\psi_t^i &= -a_i^R(\psi_t^0)\,\hat{\mathrm{d}}t, \qquad i = 1,\dots,N. \end{aligned} \tag{4.18}$$

Remark 4.4.6. Note, that unlike the deterministic BGK model (i.e. $g_k = 0$, $k = 1,\dots,d$), the stochastic case is not time homogeneous: $\varphi_{s,t}^R \neq \varphi_{0,t-s}^R$.

Proposition 4.4.7. *Let $R > 0$. If $X_0 \in C^{3,\eta}(\mathbb{T}^N\times\mathbb{R})$ almost surely,*[3] *there exists a unique strong solution to (4.16) which is a continuous $C^{3,\vartheta}$-semimartingale for some*

[2] For notational simplicity we write $G^{R,2}$ as an abbreviation for $\left(G^R\right)^2$ and similarly $g_k^{R,2}$ instead of $\left(g_k^R\right)^2$.

[3] $\eta > 0$ is the Hölder exponent from Section 4.2.

$\vartheta > 0$, *i.e. it satisfies* (4.16) *in the following sense*

$$\begin{aligned} X(t,x,\xi;s) = X_0(x,\xi) - a^R(\xi)\cdot\int_s^t \nabla X(r,x,\xi;s)\,\mathrm{d}r \\ - \sum_{k=1}^d g_k^R(x,\xi)\int_s^t \partial_\xi X(r,x,\xi;s)\circ \mathrm{d}\beta_k(r) \\ + \frac{1}{4}\partial_\xi G^{R,2}(x,\xi)\int_s^t \partial_\xi X(r,x,\xi;s)\,\mathrm{d}r, \end{aligned}$$

Moreover, it is represented by

$$X(t,x,\xi;s) = X_0\big(\psi_{s,t}^R(x,\xi)\big).$$

Proof. The above representation formula corresponds to (4.12). It can be shown in a straightforward manner using the Itô-Wentzell formula (see [50, Theorem 6.1.9]). □

It is obvious, that the domain of definition of the solution operator to (4.16), hereafter denoted by $\mathcal{S}^R$, can be extended to more general functions which do not necessarily fulfil the assumptions of Proposition 4.4.7. In this case, we define consistently

$$\mathcal{S}^R(t,s)X_0 = X_0\big(\psi_{s,t}^R(x,\xi)\big), \qquad 0\le s\le t\le T.$$

Since diffeomorphisms preserve sets of measure zero the above is well defined also if X_0 is only defined almost everywhere. The resulting process cannot be a strong solution to (4.16), however, as it will be seen in Corollary 4.4.9 it can still satisfy (4.16) in a weak sense. In the following proposition we establish basic properties of the operator $\mathcal{S}^R$.

Proposition 4.4.8. *Let $R>0$. Let $\mathcal{S}^R = \{\mathcal{S}^R(t,s), 0\le s\le t\le T\}$ be defined as above. Then*

(i) $\mathcal{S}^R$ is a family of bounded linear operators on $L^1(\Omega\times\mathbb{T}^N\times\mathbb{R})$ having unit operator norm, i.e. for any $X_0\in L^1(\Omega\times\mathbb{T}^N\times\mathbb{R})$, $0\le s\le t\le T$,

$$\big\|\mathcal{S}^R(t,s)X_0\big\|_{L^1_{\omega,x,\xi}} \le \|X_0\|_{L^1_{\omega,x,\xi}}, \tag{4.19}$$

(ii) $\mathcal{S}^R$ verifies the semigroup law

$$\begin{aligned} &\mathcal{S}^R(t,s) = \mathcal{S}^R(t,r)\circ\mathcal{S}^R(r,s), && 0\le s\le r\le t\le T,\\ &\mathcal{S}^R(s,s) = \mathrm{Id}, && 0\le s\le T. \end{aligned}$$

Proof. Fix arbitrary $0\le s\le t\le T$. The linearity of $\mathcal{S}^R(t,s)$ follows easily from its definition. In order to prove (4.19), we will proceed in several steps. First, we make an additional assumption upon the initial condition X_0, namely,

$$X_0\in L^1(\Omega\times\mathbb{T}^N\times\mathbb{R})\cap L^\infty(\Omega\times\mathbb{T}^N\times\mathbb{R}). \tag{4.20}$$

Let us now consider a suitable smooth approximation of X_0. In particular, let (h_δ) be an approximation to the identity on $\mathbb{T}^N\times\mathbb{R}$, and (k_δ) a smooth truncation on $\mathbb{R}$, i.e. define $k_\delta(\xi) = k(\delta\xi)$, where k was defined at the beginning of this subsection. Then the regularization X_0^δ, defined in the following way

$$X_0^\delta(\omega) = \big(X_0(\omega)*h_\delta\big)k_\delta,$$

is bounded, pathwise smooth and compactly supported and

$$X_0^\delta \longrightarrow X_0 \quad \text{in} \quad L^1(\Omega\times\mathbb{T}^N\times\mathbb{R}); \qquad \|X_0^\delta\|_{L^1_{\omega,x,\xi}} \le \|X_0\|_{L^1_{\omega,x,\xi}}. \tag{4.21}$$

Furthermore, also all the partial derivatives $\partial_\xi X_0^\delta$, $\partial_{x_i} X_0^\delta$, $i=1,\dots,N$, are bounded, pathwise smooth and compactly supported.

Next, the process $X^\delta = \mathcal{S}^R(t,s)X_0^\delta$ is the unique strong solution to (4.16) or equivalently

$$\begin{aligned} \mathrm{d}X + a^R(\xi)\cdot\nabla X\,\mathrm{d}t &= -\partial_\xi X \varPhi^R\,\mathrm{d}W + \frac{1}{2}\partial_\xi\big(G^{R,2}\partial_\xi X\big)\,\mathrm{d}t,\\ X(s) &= X_0^\delta \end{aligned} \tag{4.22}$$

which follows by a similar approach as in Lemma 4.4.3. For any $x\in\mathbb{T}^N$, $\xi\in\mathbb{R}$, the above stochastic integral is a well defined martingale with zero expected value. Indeed, for each $k=1,\dots,d$, we have[4]

$$\begin{aligned} \mathbb{E}\int_s^T \big|\partial_\xi X^\delta g_k^R(x,\xi)\big|^2\,\mathrm{d}r &= C\,\mathbb{E}\int_s^T \big|\nabla_{x,\xi}X_0^\delta\big(\psi_{s,r}^R(x,\xi)\big)\cdot\partial_\xi\psi_{s,r}^R(x,\xi)\big|^2\,\mathrm{d}r\\ &\le C\,\mathbb{E}\int_s^T \big|\partial_\xi\psi_{s,r}^R(x,\xi)\big|^2\,\mathrm{d}r<\infty \end{aligned}$$

since g_k^R is bounded and the process $\partial_\xi\psi_{s,r}^R(x,\xi)$ solves a backward bilinear stochastic differential equation with bounded coefficients (see [50, Theorem 4.6.5]) and therefore possesses moments of any order which are bounded in $0\le s\le r\le T$, $x\in\mathbb{T}^N$, $\xi\in\mathbb{R}$. Nevertheless, we point out the same is not generally true without the assumption (4.20). In this case, the stochastic integral can happen to be a local martingale only, which would significantly complicate the subsequent steps.

We intend to integrate the equation (4.22) with respect to the variables ω,x,ξ and expect the stochastic integral to vanish. Towards this end, it is needed to verify the interchange of integrals with respect to x, ξ and the stochastic one. We make use of the stochastic Fubini theorem [15, Theorem 4.18]. In order to verify its assumptions, the following quantity

$$\begin{aligned} &\int_{\mathbb{T}^N}\int_{\mathbb{R}}\bigg(\mathbb{E}\int_s^T \big|\partial_\xi X^\delta g_k^R(x,\xi)\big|^2\,\mathrm{d}r\bigg)^{\frac{1}{2}}\mathrm{d}\xi\,\mathrm{d}x\\ &\quad=\int_{\mathbb{T}^N}\int_{\mathbb{R}}|g_k^R(x,\xi)|\bigg(\mathbb{E}\int_s^T \big|\nabla_{x,\xi}X_0^\delta\big(\psi_{s,r}^R(x,\xi)\big)\cdot\partial_\xi\psi_{s,r}^R(x,\xi)\big|^2\,\mathrm{d}r\bigg)^{\frac{1}{2}}\mathrm{d}\xi\,\mathrm{d}x \end{aligned}$$

should be finite. Recall that g_k^R, $k=1,\dots,d$, are bounded and the moments of $\partial_\xi\psi_{s,r}^R(x,\xi)$ are finite and bounded in s, r, x, ξ. Thus, since $\nabla_{x,\xi}X_0^\delta$ is bounded and pathwise compactly supported it is sufficient to show that so does $\nabla_{x,\xi}X_0^\delta\big(\psi_{s,r}^R(x,\xi)\big)$. However, this fact follows immediately from the growth control on the stochastic flow ψ^R. Indeed, all the assertions of [50, Section 4.5], in particular Exercise 4.5.9 and 4.5.10, can be modified in order to obtain corresponding results for the component $\psi_{s,r}^{R,0}$ only. Hence, for any $\eta\in(0,1)$, we have uniformly in s, r, x, $\mathbb{P}$-a.s.,

$$\lim_{|\xi|\to\infty}\frac{|\psi_{s,r}^{R,0}(x,\xi)|}{(1+|\xi|)^{1+\eta}}=0,\qquad \lim_{|\xi|\to\infty}\frac{(1+|\xi|)^\eta}{1+|\psi_{s,r}^{R,0}(x,\xi)|}=0.$$

[4]By $\nabla_{x,\xi}$ we denote the gradient with respect to the variables x, ξ.

Consequently, it yields: for any fixed $L>0$, there exists $l>0$ such that if $|\xi|>l$ then it holds uniformly in s, r, x, $\mathbb{P}$-a.s.,

$$(1+|\xi|)^\eta \leq L(1+|\psi^{R,0}_{s,r}(x,\xi)|).$$

The support of X_0^δ as well as $\nabla_{x,\xi}X_0^\delta$ in the variable ξ is included in $[-\frac{1}{\delta},\frac{1}{\delta}]$. Therefore, if in addition $(1+|\xi|)^\eta > L(1+\frac{1}{\delta})$ then $|\psi^{R,0}_{s,r}(x,\xi)|>\frac{1}{\delta}$ for all s, r, x, $\mathbb{P}$-a.s., and accordingly $\nabla_{x,\xi}X_0^\delta\big(\psi^R_{s,r}(x,\xi)\big)=0$ for all s, r, x, $\mathbb{P}$-a.s.. As a consequence, the stochastic Fubini theorem can be applied.

Therefore, integrating the equation (4.22) with respect to ω,x,ξ yields

$$\begin{aligned}\mathbb{E}\int_{\mathbb{T}^N}\int_{\mathbb{R}} X^\delta(t,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x+\mathbb{E}\int_s^t\int_{\mathbb{R}} a^R(\xi)\cdot\int_{\mathbb{T}^N}\nabla X^\delta(r,x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}r\\
=\mathbb{E}\int_{\mathbb{T}^N}\int_{\mathbb{R}} X_0^\delta\,\mathrm{d}\xi\,\mathrm{d}x+\frac{1}{2}\mathbb{E}\int_s^t\int_{\mathbb{T}^N}\int_{\mathbb{R}}\partial_\xi\big(G^{R,2}(x,\xi)\partial_\xi X^\delta(r,x,\xi)\big)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}r\end{aligned}$$

where the second term on the left hand side vanishes due to periodic boundary conditions and the second one on the right hand side due to the compact support of $G^{R,2}$ in ξ. Hence we obtain

$$\mathbb{E}\int_{\mathbb{T}^N}\int_{\mathbb{R}}\mathcal{S}^R(t,s)X_0^\delta\,\mathrm{d}\xi\,\mathrm{d}x=\mathbb{E}\int_{\mathbb{T}^N}\int_{\mathbb{R}}X_0^\delta\,\mathrm{d}\xi\,\mathrm{d}x$$

where the integrals on both sides are finite. Note, that if X_0^δ is nonnegative (nonpositive) then also $\mathcal{S}^R(t,s)X_0^\delta$ stays nonnegative (nonpositive). Therefore,

$$\big(\mathcal{S}^R(t,s)X_0^\delta\big)^+=\mathcal{S}^R(t,s)(X_0^\delta)^+,\qquad \big(\mathcal{S}^R(t,s)X_0^\delta\big)^-=\mathcal{S}^R(t,s)(X_0^\delta)^-,$$

and by splitting the initial data into positive and negative part we obtain that (4.19) is satisfied with equality in this case.

In addition to (4.21), also the convergence $\mathcal{S}^R(t,s)X_0^\delta\to\mathcal{S}^R(t,s)X_0$ holds true in $L^1(\Omega\times\mathbb{T}^N\times\mathbb{R})$. Indeed, let us fix $\delta_1,\,\delta_2\in(0,1)$. Then (4.19) is also fulfilled by $X_0^{\delta_1}-X_0^{\delta_2}$ hence the set $\{\mathcal{S}^R(t,s)X_0^\delta;\,\delta\in(0,1)\}$ is Cauchy in $L^1(\Omega\times\mathbb{T}^N\times\mathbb{R})$ and the limit is necessarily $\mathcal{S}^R(t,s)X_0$ since diffeomorphisms preserve sets of zero measure. Finally, application of the Fatou lemma gives (4.19) for X_0.

As the next step, we avoid the hypothesis (4.20). Let $X_0\in L^1(\Omega\times\mathbb{T}^N\times\mathbb{R})$ and consider the following approximations

$$X_0^n=X_0\,\mathbf{1}_{|X_0|\leq n},\qquad n\in\mathbb{N}.$$

Then clearly

$$X_0^n\longrightarrow X_0\quad\text{in}\quad L^1(\Omega\times\mathbb{T}^N\times\mathbb{R}),\qquad \|X_0^n\|_{L^1_{\omega,x,\xi}}\leq\|X_0\|_{L^1_{\omega,x,\xi}}$$

and $X_0^n\in L^\infty(\Omega\times\mathbb{T}^N\times\mathbb{R})$ hence the estimate (4.19) is valid for all X_0^n. As above, it is possible to show that $\mathcal{S}^R(t,s)X_0^n\to\mathcal{S}^R(t,s)X_0$ in $L^1(\Omega\times\mathbb{T}^N\times\mathbb{R})$ and by the lower semicontinuity of the norm we obtain the claim.

Finally, item (ii) can be shown by the flow property of ψ:

$$\mathcal{S}^R(t,r)\circ\mathcal{S}^R(r,s)X_0=X_0\big(\psi^R_{s,r}\big(\psi^R_{r,t}(x,\xi)\big)\big)=X_0\big(\psi^R_{s,t}(x,\xi)\big)=\mathcal{S}^R(t,s)X_0.$$

□

Corollary 4.4.9. *Let $R > 0$. For any $\mathscr{F}_s \otimes \mathcal{B}(\mathbb{T}^N) \otimes \mathcal{B}(\mathbb{R})$-measurable initial datum $X_0 \in L^\infty(\Omega \times \mathbb{T}^N \times \mathbb{R})$ there exists a unique $X \in L^\infty_{\mathcal{P}_s}\big(\Omega \times [s,T] \times \mathbb{T}^N \times \mathbb{R}\big)$ that is a weak solution to (4.22), i.e. the following holds true for any $\phi \in C^\infty_c(\mathbb{T}^N \times \mathbb{R})$, a.e. $t \in [s,T]$, $\mathbb{P}$-a.s.,*

$$\begin{aligned}\big\langle X(t), \phi\big\rangle &= \big\langle X_0, \phi\big\rangle + \int_s^t \big\langle X(r), a^R \cdot \nabla\phi\big\rangle \,\mathrm{d}r \\ &+ \sum_{k=1}^d \int_s^t \big\langle X(r), \partial_\xi(g_k^R \phi)\big\rangle \,\mathrm{d}\beta_k(r) + \frac{1}{2}\int_s^t \big\langle X(r), \partial_\xi(G^{R,2}\partial_\xi\phi)\big\rangle \,\mathrm{d}r.\end{aligned} \tag{4.23}$$

Furthermore, it is represented by $X = \mathcal{S}^R(t,s)X_0$.

Proof. Let us start with the proof of uniqueness. Due to linearity, it is enough to prove that any L^∞-weak solution to (4.22) starting from the origin $X_0 = 0$ vanishes identically. Let X be such a solution. First, let (h_τ) be a symmetric approximation to the identity on $\mathbb{T}^N \times \mathbb{R}$ and test (4.22) by $\phi(x,\xi) = h_\tau(y-x, \zeta-\xi)$. (Here, we employ the parameter τ in order to distinguish from the regularization defined in Proposition 4.4.8, which will also be used in this proof.) Then $X^\tau(t) := X(t) * h_\tau$, for a.e. $t \in [s,T]$, satisfies

$$\begin{aligned}X^\tau(t,y,\zeta) = &-\int_s^t \big[a^R \cdot \nabla X(r)\big]^\tau(y,\zeta)\,\mathrm{d}r - \sum_{k=1}^d \int_s^t \big[\partial_\xi X(r) g_k^R\big]^\tau(y,\zeta)\,\mathrm{d}\beta_k(r) \\ &+ \frac{1}{2}\int_s^t \big[\partial_\xi\big(G^{R,2}\partial_\xi X(r)\big)\big]^\tau(y,\zeta)\,\mathrm{d}r\end{aligned}$$

hence is smooth in (y,ζ) and can be extended to become continuous on $[s,T]$. Now, we will argue as in [23, Theorem 20] and make use of the stochastic flow φ^R. From the Itô-Wentzell formula for the Itô integral [50, Theorem 3.3.1] we deduce

$$\begin{aligned}X^\tau\big(t, \varphi^R_{s,t}(\tilde y,\tilde\zeta)\big) = &-\int_s^t \big[a^R \cdot \nabla X(r)\big]^\tau\big(\varphi^R_{s,r}(\tilde y,\tilde\zeta)\big)\,\mathrm{d}r \\ &- \sum_{k=1}^d \int_s^t \big[\partial_\xi X(r) g_k^R\big]^\tau\big(\varphi^R_{s,r}(\tilde y,\tilde\zeta)\big)\,\mathrm{d}\beta_k(r) \\ &+ \frac{1}{2}\int_s^t \big[\partial_\xi\big(G^{R,2}\partial_\xi X(r)\big)\big]^\tau\big(\varphi^R_{s,r}(\tilde y,\tilde\zeta)\big)\,\mathrm{d}r \\ &+ \int_s^t \nabla X^\tau\big(r, \varphi^R_{s,r}(\tilde y,\tilde\zeta)\big) \cdot a^R\big(\varphi^{R,0}_{s,r}(\tilde y,\tilde\zeta)\big)\,\mathrm{d}r \\ &+ \sum_{k=1}^d \int_s^t \partial_\xi X^\tau\big(r, \varphi^R_{s,r}(\tilde y,\tilde\zeta)\big) g_k^R\big(\varphi^R_{s,r}(\tilde y,\tilde\zeta)\big)\,\mathrm{d}\beta_k(r) \\ &+ \frac{1}{2}\int_s^t \partial^2_\xi X^\tau\big(r, \varphi^R_{s,r}(\tilde y,\tilde\zeta)\big) G^{R,2}\big(\varphi^R_{s,r}(\tilde y,\tilde\zeta)\big)\,\mathrm{d}r \\ &- \sum_{k=1}^d \int_s^t \partial_\xi\big[\partial_\xi X(r) g_k^R\big]^\tau\big(\varphi^R_{s,r}(\tilde y,\tilde\zeta)\big) g_k^R\big(\varphi^R_{s,r}(\tilde y,\tilde\zeta)\big)\,\mathrm{d}r \\ = {}& \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3 + \mathrm{J}_4 + \mathrm{J}_5 + \mathrm{J}_6 + \mathrm{J}_7.\end{aligned}$$

As the next step, we intend to show that $\mathrm{J}_1 + \mathrm{J}_4 \to 0$, $\mathrm{J}_2 + \mathrm{J}_5 \to 0$, and $\mathrm{J}_3 + \mathrm{J}_6 + \mathrm{J}_7 \to 0$ in $\mathcal{D}'(\mathbb{T}^N \times \mathbb{R})$, $\mathbb{P}$-a.s., as $\tau \to 0$. Remark, that unlike [23], working with the

Stratonovich form of (4.22) would not bring any simplifications here. To be more precise, the Stratonovich version of the Itô-Wentzell formula (see [50, Theorem 3.3.2]) is close to the classical differential rule formula for composite functions hence any correction terms (as $\mathrm{J}_6, \mathrm{J}_7$ in the Itô version) are not necessary; however, due to the dependence on x, ξ of the coefficients g_k^R, the corresponding Stratonovich integrals would not cancel and therefore in order to guarantee their convergence to zero, one would need to control the correction terms $\mathrm{J}_6, \mathrm{J}_7$ anyway.

Let us proceed with the proof of the above sketched convergence. Towards this end, we employ repeatedly the arguments of the commutation lemma of DiPerna and Lions [17, Lemma II.1]. In particular, in the case of $\mathrm{J}_1+\mathrm{J}_4$ we obtain for a.e. $r \in [s,t]$, $\mathbb{P}$-a.s., that

$$a^R \cdot \nabla X^\tau(r) - \big[a^R \cdot \nabla X(r)\big]^\tau \longrightarrow 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{T}^N \times \mathbb{R}). \tag{4.24}$$

Indeed, since

$$\begin{aligned} &a^R(\xi) \cdot \nabla X^\tau(r,x,\xi) - \big[a^R \cdot \nabla X(r)\big]^\tau(x,\xi) \\ &\quad = \int_{\mathbb{T}^N}\int_{\mathbb{R}} X(r,y,\zeta)\big[a^R(\xi)-a^R(\zeta)\big] \cdot \nabla h_\tau(x-y,\xi-\zeta)\mathrm{d}\zeta\mathrm{d}y \end{aligned}$$

and $\tau|\nabla h_\tau|(\cdot) \leq C h_{2\tau}(\cdot)$, we obtain the following bound by standard estimates on convolutions : for any $\phi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$

$$\begin{aligned} &\Big|\Big\langle a^R \cdot \nabla X^\tau(r) - \big[a^R \cdot \nabla X(r)\big]^\tau, \phi\Big\rangle\Big| \\ &\qquad\qquad \leq C\big\|a^R\big\|_{W^{1,\infty}(\mathbb{R})}\|X(r)\|_{L^p(K_\phi)}\|\phi\|_{L^q(\mathbb{T}^N\times\mathbb{R})}, \end{aligned}$$

where $K_\phi \subset \mathbb{T}^N \times \mathbb{R}$ is a suitable compact set and $p, q \in [1,\infty]$ are arbitrary conjugate exponents. As a consequence, it is sufficient to consider $X(r)$ continuous in (x,ξ) as the general case can be concluded by a density argument. We have

$$\begin{aligned} &\int_{\mathbb{T}^N}\int_{\mathbb{R}} X(r,y,\zeta)\big[a^R(\xi)-a^R(\zeta)\big] \cdot \nabla h_\tau(x-y,\xi-\zeta)\mathrm{d}\zeta\mathrm{d}y \\ &= \int_{\mathbb{T}^N}\int_{\mathbb{R}}\int_0^1 X(r,y,\zeta)\mathrm{D}a^R\big(\zeta+\sigma(\xi-\zeta)\big)(\xi-\zeta) \cdot \nabla h_\tau(x-y,\xi-\zeta)\mathrm{d}\sigma\mathrm{d}\zeta\mathrm{d}y \\ &= \int_{\mathbb{T}^N}\int_{\mathbb{R}}\int_0^1 X\big(r,x-\tau\tilde{y},\xi-\tau\tilde{\zeta}\big)\mathrm{D}a^R\big(\xi-(1-\sigma)\tau\tilde{\zeta}\big)\tilde{\zeta} \cdot \nabla h(\tilde{y},\tilde{\zeta})\mathrm{d}\sigma\mathrm{d}\tilde{\zeta}\mathrm{d}\tilde{y} \\ &\longrightarrow X(r,x,\xi)\mathrm{D}a^R(\xi) \cdot \int_{\mathbb{T}^N}\int_{\mathbb{R}} \tilde{\zeta}\,\nabla h(\tilde{y},\tilde{\zeta})\mathrm{d}\tilde{\zeta}\mathrm{d}\tilde{y} = 0 \end{aligned}$$

hence (4.24) follows by the dominated convergence theorem. Moreover, we deduce also that for a.e. $r \in [s,t]$, $\mathbb{P}$-a.s.,

$$a^R\big(\varphi_{s,r}^{R,0}\big) \cdot \nabla X^\tau\big(r,\varphi_{s,r}^R\big) - \big[a^R \cdot \nabla X(r)\big]^\tau\big(\varphi_{s,r}^R\big) \longrightarrow 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{T}^N \times \mathbb{R}). \tag{4.25}$$

It can be seen by using the change of variables formula: let $\mathrm{J}\psi^R_{s,r}$ denote the Jacobian of the inverse flow $\psi^R_{s,r}$, then

$$
\begin{aligned}
&\Big|\Big\langle a^R\big(\varphi^{R,0}_{s,r}\big)\cdot\nabla X^\tau\big(r,\varphi^R_{s,r}\big)-\big[a^R\cdot\nabla X(r)\big]^\tau\big(\varphi^R_{s,r}\big),\phi\Big\rangle\Big|\\
&=\Big|\Big\langle a^R\cdot\nabla X^\tau(r)-\big[a^R\cdot\nabla X(r)\big]^\tau,\phi\big(\psi^R_{s,r}\big)\big|\mathrm{J}\psi^R_{s,r}\big|\Big\rangle\Big|\\
&\leq C\big\|a^R\big\|_{W^{1,\infty}(\mathbb{R})}\|X(r)\|_{L^p(K)}\big\|\phi\big(\psi^R_{s,r}\big)\mathrm{J}\psi^R_{s,r}\big\|_{L^q(K)}\\
&\leq C\big\|a^R\big\|_{W^{1,\infty}(\mathbb{R})}\operatorname*{ess\,sup}_{s\leq r\leq T}\|X(r)\|_{L^p(K)}\|\phi\|_{L^\infty(K)}\sup_{s\leq r\leq T}\big\|\mathrm{J}\psi^R_{s,r}\big\|_{L^q(K)}<\infty,
\end{aligned}
$$

which holds for a suitably chosen compact set $K\subset\mathbb{T}^N\times\mathbb{R}$ as $\phi(\psi^R_{s,r})$ is compactly supported in $\mathbb{T}^N\times\mathbb{R}$ and any conjugate exponents $p,q\in[1,\infty]$. The estimate of $\sup_{s\leq r\leq T}\|\mathrm{J}\psi^R_{s,r}\|_{L^q(K)}$ is an immediate consequence of the fact that for almost every $\omega\in\Omega$ the mapping $(r,x,\xi)\mapsto\mathrm{D}\psi^R_{s,r}(\omega,x,\xi)$ is continuous due to the properties of stochastic flows (see Subsection 4.3.2, (vi)) and therefore $(r,x,\xi)\mapsto\mathrm{J}\psi^R_{s,r}(\omega,x,\xi)$ is bounded on the given compact set $[s,T]\times K$. Having this bound in hand, we infer (4.25) by using density again. Accordingly, the almost sure convergence $\mathrm{J}_1+\mathrm{J}_4\to0$ in $\mathcal{D}'(\mathbb{T}^N\times\mathbb{R})$ follows by the dominated convergence theorem.

In order to pass to the limit in the case of $\mathrm{J}_2+\mathrm{J}_5$, we employ the same approach as above so we will only write the main points of the proof. We obtain

$$
\begin{aligned}
&\Big|\Big\langle g^R_k\big(\varphi^R_{s,r}\big)\,\partial_\xi X^\tau\big(r,\varphi^R_{s,r}\big)-\big[g^R_k\,\partial_\xi X(r)\big]^\tau\big(\varphi^R_{s,r}\big),\phi\Big\rangle\Big|\\
&\qquad\leq C\big\|g^R_k\big\|_{W^{1,\infty}(\mathbb{R})}\operatorname*{ess\,sup}_{s\leq r\leq T}\|X(r)\|_{L^p(K)}\|\phi\|_{L^\infty(K)}\sup_{s\leq r\leq T}\big\|\mathrm{J}\psi^R_{s,r}\big\|_{L^q(K)}
\end{aligned}
$$

hence for a.e. $r\in[s,T]$, $\mathbb{P}$-a.s.,

$$
g^R_k\big(\varphi^R_{s,r}\big)\,\partial_\xi X^\tau\big(r,\varphi^R_{s,r}\big)-\big[g^R_k\,\partial_\xi X(r)\big]^\tau\big(\varphi^R_{s,r}\big)\longrightarrow0\quad\text{in}\quad\mathcal{D}'(\mathbb{T}^N\times\mathbb{R})
$$

and accordingly we conclude by the dominated convergence theorem for stochastic integrals [67, Theorem 32] that $\mathbb{P}$-a.s. (up to subsequences) $\mathrm{J}_2+\mathrm{J}_5\to0$ in $\mathcal{D}'(\mathbb{T}^N\times\mathbb{R})$.

Now, it remains to verify the convergence of $\mathrm{J}_3+\mathrm{J}_6+\mathrm{J}_7$. As the first step, we will show that for a.e. $r\in[s,T]$, $\mathbb{P}$-a.s., in $\mathcal{D}'(\mathbb{T}^N\times\mathbb{R})$

$$
\frac12\big[\partial_\xi\big(g^{R,2}_k\partial_\xi X(r)\big)\big]^\tau+\frac12\partial^2_{\xi\xi}X^\tau(r)g^{R,2}_k-\partial_\xi\big[\partial_\xi X(r)g^R_k\big]^\tau g^R_k\longrightarrow0. \tag{4.26}
$$

Towards this end, we observe

$$
\begin{aligned}
\frac12\big[\partial_\xi\big(g^{R,2}_k\partial_\xi X(r)\big)\big]^\tau(x,\xi)&=\frac12\big\langle\partial_\zeta X(r)g^{R,2}_k,\partial_\xi h_\tau(x-\cdot,\xi-\cdot)\big\rangle,\\
\frac12\partial^2_{\xi\xi}X^\tau(r,x,\xi)g^{R,2}_k(x,\xi)&=\frac12\big\langle\partial_\zeta X(r),\partial_\xi h_\tau(x-\cdot,\xi-\cdot)\big\rangle g^{R,2}_k(x,\xi),\\
-\partial_\xi\big[\partial_\xi X(r)g^R_k\big]^\tau(x,\xi)\,g^R_k(x,\xi)&=-\big\langle\partial_\zeta X(r)g^R_k,\partial_\xi h_\tau(x-\cdot,\xi-\cdot)\big\rangle g^R_k(x,\xi),
\end{aligned}
$$

and hence the left hand side of (4.26) evaluated at (x,ξ) is equal to

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{T}^N}\int_{\mathbb{R}}\partial_\zeta X(r,y,\zeta)\big(g_k^R(y,\zeta)-g_k^R(x,\xi)\big)^2\partial_\xi h_\tau(x-y,\xi-\zeta)\mathrm{d}\zeta\mathrm{d}y\\
&=-\int_{\mathbb{T}^N}\int_{\mathbb{R}}X(r,y,\zeta)\big(g_k^R(y,\zeta)-g_k^R(x,\xi)\big)\partial_\zeta g_k^R(y,\zeta)\partial_\xi h_\tau(x-y,\xi-\zeta)\mathrm{d}\zeta\mathrm{d}y\\
&\quad+\frac{1}{2}\int_{\mathbb{T}^N}\int_{\mathbb{R}}X(r,y,\zeta)\big(g_k^R(y,\zeta)-g_k^R(x,\xi)\big)^2\partial^2_{\xi\xi}h_\tau(x-y,\xi-\zeta)\mathrm{d}\zeta\mathrm{d}y\\
&=\mathrm{I}_1(x,\xi)+\mathrm{I}_2(x,\xi).
\end{aligned}$$

Next, we proceed as in the case of $\mathrm{J}_1+\mathrm{J}_4$. We obtain

$$\big|\langle \mathrm{I}_1+\mathrm{I}_2,\phi\rangle\big|\leq C\|g_k^R\|^2_{W^{1,\infty}(\mathbb{T}^N\times\mathbb{R})}\|X(r)\|_{L^p(K_\phi)}\|\phi\|_{L^q(\mathbb{T}^N\times\mathbb{R})}$$

which holds true for a suitable compact set $K_\phi\subset\mathbb{T}^N\times\mathbb{R}$ and arbitrary conjugate exponents $p,q\in[1,\infty]$ and in the case of $X(r)$ continuous in (x,ξ)

$$\begin{aligned}
\mathrm{I}_1(x,\xi)&\longrightarrow -X(r,x,\xi)\big(\partial_\xi g_k^R(x,\xi)\big)^2,\\
\mathrm{I}_2(x,\xi)&\longrightarrow X(r,x,\xi)\big(\partial_\xi g_k^R(x,\xi)\big)^2,
\end{aligned}$$

which yields (4.26) by the dominated convergence theorem and density. As the next step, we conclude that

$$\begin{aligned}
&\Big|\Big\langle \mathrm{I}_1(\varphi^R_{s,r})+\mathrm{I}_2(\varphi^R_{s,r}),\phi\Big\rangle\Big|\\
&\leq C\|g_k^R\|^2_{W^{1,\infty}(\mathbb{T}^N\times\mathbb{R})}\operatorname*{ess\,sup}_{s\leq r\leq T}\|X(r)\|_{L^p(K)}\|\phi\|_{L^\infty(K)}\sup_{s\leq r\leq T}\big\|\mathrm{J}\psi^R_{s,r}\big\|_{L^q(K)}
\end{aligned}$$

and consequently for a.e. $r\in[s,T]$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\frac{1}{2}\big[\partial_\xi\big(g_k^{R,2}\partial_\xi X(r)\big)\big]^\tau(\varphi^R_{s,r})+\frac{1}{2}\partial^2_{\xi\xi}X^\tau(r,\varphi^R_{s,r})g_k^{R,2}(\varphi^R_{s,r})\\
&\qquad-\partial_\xi\big[\partial_\xi X(r)g_k^R\big]^\tau(\varphi^R_{s,r})g_k^R(\varphi^R_{s,r})\longrightarrow 0\quad\text{in}\quad\mathcal{D}'(\mathbb{T}^N\times\mathbb{R}).
\end{aligned}$$

Therefore, the desired convergence of $\mathrm{J}_3+\mathrm{J}_6+\mathrm{J}_7$ is verified.

Finally, since it holds true for a.e. $t\in[s,T]$ that

$$X^\tau(t)\xrightarrow{w^*}X(t)\qquad\text{in}\qquad L^\infty(\mathbb{T}^N\times\mathbb{R}),\ \mathbb{P}\text{-a.s.},$$

we obtain for any $\phi\in C_c^\infty(\mathbb{T}^N\times\mathbb{R})$

$$\begin{aligned}
\Big\langle X(t,\varphi^R_{s,t}),\phi\Big\rangle&=\Big\langle X(t),\phi(\psi^R_{s,t})\big|\mathrm{J}\psi^R_{s,t}\big|\Big\rangle=\lim_{\tau\to0}\Big\langle X^\tau(t),\phi(\psi^R_{s,t})\big|\mathrm{J}\psi^R_{s,t}\big|\Big\rangle\\
&=\lim_{\tau\to0}\Big\langle X^\tau(t,\varphi^R_{s,t}),\phi\Big\rangle=0
\end{aligned}$$

hence $X=0$ since $\varphi^R_{s,t}$ is a bijection and the proof of uniqueness is complete.

The proof of the explicit formula for X follows by employing the regularization X_0^δ as in the proof of Proposition 4.4.8. The process $X^\delta=\mathcal{S}^R(t,s)X_0^\delta$ is the unique strong solution to (4.16) or equivalently (4.22) by using a similar approach as in Lemma 4.4.3.

Consequently, it satisfies for all $\phi\in C_c^\infty(\mathbb{T}^N\times\mathbb{R})$

$$\begin{aligned}\big\langle X^\delta(t),\phi\big\rangle&=\big\langle X_0^\delta,\phi\big\rangle+\int_s^t\big\langle X^\delta(r),a^R(\xi)\cdot\nabla\phi\big\rangle\,\mathrm{d}r\\&+\sum_{k=1}^d\int_s^t\big\langle X^\delta(r),\partial_\xi(g_k^R\phi)\big\rangle\,\mathrm{d}\beta_k(r)+\frac12\int_s^t\big\langle X^\delta(r),\partial_\xi(G^{R,2}\partial_\xi\phi)\big\rangle\,\mathrm{d}r.\end{aligned}$$

Now, it only remains to take the limit as $\delta\to0$. As $X_0^\delta\to X_0$ for a.e. ω,x,ξ we have $X^\delta=\mathcal{S}^R(t,s)X_0^\delta\to\mathcal{S}^R(t,s)X_0=X$ for a.e. ω,x,ξ and every $t\in[s,T]$. Therefore, the convergence in all the terms apart from the stochastic one follows directly by the dominated convergence theorem. For the case of stochastic integral we can apply the dominated convergence theorem for stochastic integrals. Since it holds

$$\big\langle X^\delta(r),\partial_\xi(g_k^R\phi)\big\rangle\longrightarrow\big\langle X(r),\partial_\xi(g_k^R\phi)\big\rangle,\qquad\text{a.e. }(\omega,\,r)\in\Omega\times[s,T]$$

and, setting $K=\operatorname{supp}\phi\subset\mathbb{T}^N\times\mathbb{R}$,

$$\big|\big\langle X^\delta(r),\partial_\xi(g_k^R\phi)\big\rangle\big|\le C\int_K\big|X_0^\delta\big(\psi_{s,r}^R(x,\xi)\big)\big|\,\mathrm{d}\xi\,\mathrm{d}x\le C,$$

where the constant C does not depend on δ due to the fact that

$$\|X_0^\delta\|_{L^\infty_{\omega,x,\xi}}\le\|X_0\|_{L^\infty_{\omega,x,\xi}}.$$

Thus, we deduce (up to subsequences) the almost sure convergence of the stochastic integrals. Furthermore, $\mathcal{S}^R(t,s)X_0$ is exactly the representative (in t) of the unique weak solution of (4.22) that satisfies (4.23) for all $t\in[s,T]$, in particular, $t\mapsto\langle\mathcal{S}^R(t,s)X_0,\phi\rangle$ is a continuous $(\mathscr{F}_t)_{t\ge s}$-semimartingale for any $\phi\in C_c^\infty(\mathbb{T}^N\times\mathbb{R})$. $\square$

As the next step, we derive the existence of a unique weak solution to (4.14) which can be equivalently rewritten as

$$\begin{aligned}\mathrm{d}X+a(\xi)\cdot\nabla X\,\mathrm{d}t&=-\partial_\xi X\varPhi\,\mathrm{d}W+\frac12\partial_\xi(G^2\partial_\xi X)\,\mathrm{d}t,\\X(s)&=X_0\end{aligned}\tag{4.27}$$

due to Lemma 4.4.3. With regard to the definition of the truncated coefficients, let us define

$$\tau^R(s,x,\xi)=\inf\big\{t\ge s;\,|\varphi_{s,t}^{R,0}(x,\xi)|>R\big\}$$

(with the convention $\inf\emptyset=T$). Clearly, for any $s\in[0,T]$, $x\in\mathbb{T}^N$, $\xi\in\mathbb{R}$, $\tau^R(s,x,\xi)$ is a stopping time with respect to the filtration $(\mathscr{F}_t)_{t\ge s}$. Nevertheless, it can be shown that the blow-up cannot occur in a finite time and therefore

$$\sup_{R>0}\tau^R(s,x,\xi)=T,\qquad\mathbb{P}\text{-a.s., }s\in[0,T],\,x\in\mathbb{T}^N,\,\xi\in\mathbb{R}.$$

Indeed, for any $R>0$, the process $\varphi^{R,0}$ satisfies the Itô equation

$$\mathrm{d}\varphi_t^{R,0}=\sum_{k=1}^d g_k^R(\varphi_t^R)\,\mathrm{d}\beta_k(t)$$

where all the coefficients g_k^R satisfy the linear growth estimate (4.4) that is independent of R and x and therefore the claim follows by a standard estimation technique for SDEs. Moreover, if $R'>R$ then due to uniqueness $\tau^{R'}(s,x,\xi)\geq\tau^R(s,x,\xi)$ and $\mathcal{S}^{R'}(t,s)X_0=\mathcal{S}^R(t,s)X_0$ on $[0,\tau^R(s,x,\xi)]$. As a consequence, the pointwise limit

$$\big[\mathcal{S}(t,s)X_0\big](\omega,x,\xi):=\lim_{R\to\infty}\big[\mathcal{S}^R(t,s)X_0\big](\omega,x,\xi),\qquad 0\leq s\leq t\leq T,$$

exists almost surely and we obtain the following result.

Corollary 4.4.10. *The family $\mathcal{S}=\{\mathcal{S}(t,s),\,0\leq s\leq t\leq T\}$ consists of bounded linear operators on $L^1(\Omega\times\mathbb{T}^N\times\mathbb{R})$ having unit operator norm, i.e. for any $X_0\in L^1(\Omega\times\mathbb{T}^N\times\mathbb{R})$, $0\leq s\leq t\leq T$,*

$$\big\|\mathcal{S}(t,s)X_0\big\|_{L^1_{\omega,x,\xi}}\leq\|X_0\|_{L^1_{\omega,x,\xi}}.$$

Furthermore, for any $\mathscr{F}_s\otimes\mathcal{B}(\mathbb{T}^N)\otimes\mathcal{B}(\mathbb{R})$-measurable initial datum $X_0\in L^\infty(\Omega\times\mathbb{T}^N\times\mathbb{R})$ there exists a unique $X\in L^\infty_{\mathcal{P}_s}(\Omega\times[s,T]\times\mathbb{T}^N\times\mathbb{R})$ that is a weak solution to (4.27). Besides, it is represented by $X=\mathcal{S}(t,s)X_0$ and $t\mapsto\langle\mathcal{S}(t,s)X_0,\phi\rangle$ is a continuous $(\mathscr{F}_t)_{t\geq s}$-semimartingale for any $\phi\in C_c^\infty(\mathbb{T}^N\times\mathbb{R})$. Consequently, $\mathcal{S}$ verifies the semigroup law

$$\begin{aligned}\mathcal{S}(t,s)&=\mathcal{S}(t,r)\circ\mathcal{S}(r,s), && 0\leq s\leq r\leq t\leq T,\\ \mathcal{S}(s,s)&=\mathrm{Id}, && 0\leq s\leq T.\end{aligned}$$

Proof. The first part of the proof follows directly from Proposition 4.4.8 while the rest is a consequence of Corollary 4.4.9. □

Corollary 4.4.11. *For all $n\in[0,\infty)$ it holds*

$$\sup_{0\leq s\leq T}\mathbb{E}\sup_{s\leq t\leq T}\big\|\big(\mathcal{S}(t,s)\mathbf{1}_{0>\xi}-\mathbf{1}_{0>\xi}\big)(1+|\xi|)^n\big\|_{L^1_{x,\xi}}\leq C.\tag{4.28}$$

Proof. Remark, that if (4.5) is fulfilled, then for any $0\leq s\leq t\leq T$ and $x\in\mathbb{T}^N$ the process $\varphi^{R,0}_{s,t}(x,0)\equiv0$ is a solution to the first equation in (4.17) for any $R>0$. Moreover, since the solution to (4.17) is unique, we deduce

$$\varphi^{R,0}_{s,t}(x,\xi)\begin{cases}\geq0, & \text{if }\ \xi\geq0,\\ \leq0, & \text{if }\ \xi\leq0.\end{cases}$$

As a consequence, the same is valid for the inverse stochastic flow $\psi^{R,0}$ hence

$$\mathcal{S}^R(t,s)\mathbf{1}_{0>\xi}=\mathbf{1}_{0>\xi}$$

for all $R>0$ and thus the left hand side in (4.28) is zero.

In the case of (4.6), it is enough to prove the statement for any $\mathcal{S}^R$ provided the constant is independent on R. The stochastic characteristic system (4.17) rewritten in terms of Itô's integral takes the following form

$$\begin{aligned}\mathrm{d}\varphi_t^0&=\sum_{k=1}^d g_k^R(\varphi_t)\,\mathrm{d}\beta_k(t),\\ \mathrm{d}\varphi_t^i&=a_i^R(\varphi_t^0)\,\mathrm{d}t,\qquad i=1,\dots,N,\end{aligned}$$

whereas, in the case of the inverse flow, (4.18) reads

$$\begin{aligned}
\mathrm{d}\psi_t^0 &= -\sum_{k=1}^d g_k^R(\psi_t)\,\hat{\mathrm{d}}\beta_k(t),\\
\mathrm{d}\psi_t^i &= -a_i^R(\psi_t^0)\,\mathrm{d}t, \qquad i=1,\dots,N.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\mathcal{S}^R(t,s)\mathbf{1}_{0>\xi} - \mathbf{1}_{0>\xi} &= \mathbf{1}_{\sum_{k=1}^d \int_s^t g_k^R(\psi_{r,t}^R(x,\xi))\hat{\mathrm{d}}\beta_k(r)>\xi} - \mathbf{1}_{0>\xi}\\
&= \mathbf{1}_{|\xi|\leq \left|\sum_{k=1}^d \int_s^t g_k^R(\psi_{r,t}^R(x,\xi))\hat{\mathrm{d}}\beta_k(r)\right|}\\
&\leq \frac{\big(1+\big|\sum_{k=1}^d \int_s^t g_k^R(\psi_{r,t}^R(x,\xi))\hat{\mathrm{d}}\beta_k(r)\big|\big)^{n+2}}{(1+|\xi|)^{n+2}}
\end{aligned}$$

and since the fact that $\psi_{r,t}^R\circ\varphi_{s,t}^R=\varphi_{s,r}^R$ implies

$$\sum_{k=1}^d \int_s^t g_k^R(\psi_{r,t}^R(x,\xi))\,\hat{\mathrm{d}}\beta_k(r) = \sum_{k=1}^d \int_s^t g_k^R(\varphi_{s,r}^R(y,\zeta))\,\mathrm{d}\beta_k(r)$$

by setting $(x,\xi)=\varphi_{s,t}^R(y,\zeta)$, we deduce that

$$\begin{aligned}
&\mathbb{E}\sup_{s\leq t\leq T}\int_{\mathbb{T}^N}\int_{\mathbb{R}}\big|\mathcal{S}(t,s)\mathbf{1}_{0>\xi}-\mathbf{1}_{0>\xi}\big|(1+|\xi|)^n\,\mathrm{d}\xi\,\mathrm{d}x\\
&\quad\leq C + C\sup_{(y,\zeta)\in\mathbb{R}^N\times\mathbb{R}}\mathbb{E}\sup_{s\leq t\leq T}\left|\sum_{k=1}^d\int_s^t g_k^R(\varphi_{s,r}^R(y,\zeta))\,\mathrm{d}\beta_k(r)\right|^{n+2}\\
&\quad\leq C + C\sup_{(y,\zeta)\in\mathbb{R}^N\times\mathbb{R}}\mathbb{E}\left(\sum_{k=1}^d\int_s^T\big|g_k^R(\varphi_{s,r}^R(y,\zeta))\big|^2\,\mathrm{d}r\right)^{\frac{n+2}{2}} \leq C,
\end{aligned}$$

where the constant C does not depend on R and s. □

Remark 4.4.12. Let us make some comments on hypotheses (4.5), (4.6) as the proof of Corollary 4.4.11 is their only use. The main difficulty in proving (4.28) comes from the unknown structure of dependence of the stochastic flows φ^R and ψ^R on ξ in connection with the remaining variables ω,x,s,t. Although one cannot say much in general, it is possible to find some (mostly simple) examples such that (4.28) holds true even without (4.5), (4.6). If the stochastic characteristic curve is governed by a linear system of stochastic differential equation as for instance

$$\begin{aligned}
\mathrm{d}\varphi_t^0 &= \sum_{k=0}^N \big(1+\varphi_t^k\big)\,\mathrm{d}\beta_k(t),\\
\mathrm{d}\varphi_t^i &= \varphi_t^0\,\mathrm{d}t, \qquad i=1,\dots,N,
\end{aligned}$$

i.e. neither (4.5) nor (4.6) is fulfilled since $g_0(x,\xi)=1+\xi$, then both forward and backward stochastic flow are given by explicit formulas where the dependence on ξ is clear and, as a consequence, the statement of Corollary 4.4.11 remains valid.

Now, we have all in hand to complete the proof of Theorem 4.4.5.

Proof of Theorem 4.4.5. Recall, that the local densities are defined as follows

$$u^\varepsilon(t,x)=\int_{\mathbb{R}} f^\varepsilon(t,x,\xi)\,\mathrm{d}\xi=\int_{\mathbb{R}}\big(F^\varepsilon(t,x,\xi)-\mathbf{1}_{0>\xi}\big)\,\mathrm{d}\xi \tag{4.29}$$

hence the function F^ε is not integrable with respect to ξ. For the purpose of the proof it is therefore more convenient to consider the process $h^\varepsilon(t)=F^\varepsilon(t)-\mathcal{S}(t,0)\mathbf{1}_{0>\xi}$ instead and prove that it exists and is given by a suitable integral representation. Due to Corollary 4.4.10, $\mathcal{S}(t,s)\mathbf{1}_{0>\xi}$ is the unique weak solution to (4.27) hence h^ε solves

$$\begin{aligned}\mathrm{d}h^\varepsilon+a(\xi)\cdot\nabla h^\varepsilon\,\mathrm{d}t&=\frac{(\mathbf{1}_{u^\varepsilon>\xi}-\mathcal{S}(t,0)\mathbf{1}_{0>\xi})-h^\varepsilon}{\varepsilon}\,\mathrm{d}t-\partial_\xi h^\varepsilon\varPhi\,\mathrm{d}W\\&\quad-\frac{1}{2}\partial_\xi\big(G^2(-\partial_\xi h^\varepsilon)\big)\,\mathrm{d}t,\\ h^\varepsilon(0)&=\chi_{u_0^\varepsilon},\end{aligned}\tag{4.30}$$

in the sense of distributions. Then, by Lemma 4.4.3 and the weak version of Duhamel's principle, the problem (4.30) admits an equivalent integral representation

$$h^\varepsilon(t)=\mathrm{e}^{-\frac{t}{\varepsilon}}\mathcal{S}(t,0)\chi_{u_0^\varepsilon}+\frac{1}{\varepsilon}\int_0^t\mathrm{e}^{-\frac{t-s}{\varepsilon}}\mathcal{S}(t,s)\big[\mathbf{1}_{u^\varepsilon(s)>\xi}-\mathcal{S}(s,0)\mathbf{1}_{0>\xi}\big]\mathrm{d}s \tag{4.31}$$

and thus can be solved by a fixed point method. According to the identity

$$\int_{\mathbb{R}}|\mathbf{1}_{\alpha>\xi}-\mathbf{1}_{\beta>\xi}|\,\mathrm{d}\xi=|\alpha-\beta|,\qquad \alpha,\,\beta\in\mathbb{R},$$

some space of ξ-integrable functions seems to be well suited to deal with the nonlinearity term $\mathbf{1}_{u^\varepsilon>\xi}$. Let us denote $\mathscr{H}=L^\infty(0,T;L^1(\Omega\times\mathbb{T}^N\times\mathbb{R}))$ and show that the mapping

$$(\mathscr{K}g)(t)=\mathrm{e}^{-\frac{t}{\varepsilon}}\mathcal{S}(t,0)\chi_{u_0^\varepsilon}+\frac{1}{\varepsilon}\int_0^t\mathrm{e}^{-\frac{t-s}{\varepsilon}}\mathcal{S}(t,s)\big[\mathbf{1}_{v(s)>\xi}-\mathcal{S}(s,0)\mathbf{1}_{0>\xi}\big]\mathrm{d}s,$$

where the local density $v(s)=\int_{\mathbb{R}}(g(s,\xi)+\mathcal{S}(s,0)\mathbf{1}_{0>\xi}-\mathbf{1}_{0>\xi})\mathrm{d}\xi$ is defined consistently with (4.29), is a contraction on $\mathscr{H}$. Let g, g_1, $g_2\in\mathscr{H}$ with corresponding densities v, v_1, v_2. By Proposition 4.4.8, Corollary 4.4.11 and the assumptions on initial data, we arrive at

$$\begin{aligned}&\big\|(\mathscr{K}g)(t)\big\|_{L^1_{\omega,x,\xi}}\\&\quad\le\mathrm{e}^{-\frac{t}{\varepsilon}}\|\chi_{u_0^\varepsilon}\|_{L^1_{\omega,x,\xi}}+\frac{1}{\varepsilon}\int_0^t\mathrm{e}^{-\frac{t-s}{\varepsilon}}\|\mathbf{1}_{v(s)>\xi}-\mathcal{S}(s,0)\mathbf{1}_{0>\xi}\|_{L^1_{\omega,x,\xi}}\mathrm{d}s\\&\quad\le\|u_0^\varepsilon\|_{L^1_{\omega,x}}+\sup_{0\le s\le t}\Big(\|\chi_{v(s)}\|_{L^1_{\omega,x,\xi}}+\|\mathcal{S}(s,0)\mathbf{1}_{0>\xi}-\mathbf{1}_{0>\xi}\|_{L^1_{\omega,x,\xi}}\Big)\\&\quad\le C+\sup_{0\le s\le t}\|g(s)\|_{L^1_{\omega,x,\xi}},\end{aligned}$$

with a constant independent on t, hence

$$\big\|\mathscr{K}g\big\|_{L^\infty_tL^1_{\omega,x,\xi}}\le C+\|g\|_{L^\infty_tL^1_{\omega,x,\xi}}<\infty.$$

Next, we have

$$\begin{aligned}\big\|(\mathscr{K}g_1)(t)-(\mathscr{K}g_2)(t)\big\|_{L^1_{\omega,x,\xi}}&\leq\frac{1}{\varepsilon}\int_0^t\mathrm{e}^{-\frac{t-s}{\varepsilon}}\|\mathbf{1}_{v_1(s)>\xi}-\mathbf{1}_{v_2(s)>\xi}\|_{L^1_{\omega,x,\xi}}\mathrm{d}s\\&=\frac{1}{\varepsilon}\int_0^t\mathrm{e}^{-\frac{t-s}{\varepsilon}}\|v_1(s)-v_2(s)\|_{L^1_{\omega,x}}\mathrm{d}s\\&\leq\frac{1}{\varepsilon}\int_0^t\mathrm{e}^{-\frac{t-s}{\varepsilon}}\|g_1(s)-g_2(s)\|_{L^1_{\omega,x,\xi}}\mathrm{d}s,\end{aligned}$$

so

$$\big\|\mathscr{K}g_1-\mathscr{K}g_2\big\|_{L^\infty_tL^1_{\omega,x,\xi}}\leq\big(1-\mathrm{e}^{-\frac{T}{\varepsilon}}\big)\|g_1-g_2\|_{L^\infty_tL^1_{\omega,x,\xi}}$$

and according to the Banach fixed point theorem, the mapping $\mathscr{K}$ has a unique fixed point in $\mathscr{H}$. Moreover, we deduce from Corollary 4.4.10 that h^ε is measurable with respect to $\mathcal{P}\otimes\mathcal{B}(\mathbb{T}^N)\otimes\mathcal{B}(\mathbb{R})$ and therefore, according to the semigroup property of the solution operator $\mathcal{S}$, we obtain the existence of a unique weak solution to (4.3) that is expressed as (4.15) and the proof is complete. □

Remark 4.4.13. As a consequence of Corollary 4.4.10, it can be seen that the representative $h^\varepsilon(t)$ of the unique weak solution to (4.30) that is given by (4.31) satisfies: $t\mapsto\langle h^\varepsilon(t),\phi\rangle$ is a continuous $(\mathscr{F}_t)$-semimartingale for any $\phi\in C_c^\infty(\mathbb{T}^N\times\mathbb{R})$. Accordingly, $t\mapsto\langle F^\varepsilon(t),\phi\rangle$ is a continuous $(\mathscr{F}_t)$-semimartingale for any $\phi\in C_c^\infty(\mathbb{T}^N\times\mathbb{R})$ provided $F^\varepsilon(t)$ is the representative of the unique weak solution to (4.3) given by (4.15).

4.4.2 Further properties of the solution operator

In the previous subsection we showed that the family $\mathcal{S}$ consists of bounded linear operators on $L^1(\Omega\times\mathbb{T}^N\times\mathbb{R})$ with unit operator norm which was essential for the existence proof for the stochastic BGK model in Theorem 4.4.5. Nevertheless, for the proof of convergence of the BGK approximation in the next section, namely, to derive certain uniform estimates, we need to study also its behavior in other spaces. In particular, $\mathcal{S}(t,s)X_0$ is well defined if $X_0\in L^p(\Omega\times\mathbb{T}^N\times\mathbb{R})$ and we obtain the following result.

Proposition 4.4.14. *For any $p\in[2,\infty)$, the family $\mathcal{S}$ consists of bounded linear operators on $L^p(\Omega\times\mathbb{T}^N\times\mathbb{R})$ having unit operator norm. Moreover, the solution to (4.14) belongs to $L^p(\Omega;L^\infty(0,T;L^p(\mathbb{T}^N\times\mathbb{R})))$ provided $X_0\in L^p(\Omega\times\mathbb{T}^N\times\mathbb{R})$ and the following estimate holds true*

$$\sup_{0\leq s\leq T}\mathbb{E}\sup_{s\leq t\leq T}\big\|\mathcal{S}(t,s)X_0\big\|^p_{L^p_{x,\xi}}\leq C\,\|X_0\|^p_{L^p_{\omega,x,\xi}}.\tag{4.32}$$

Proof. Note, that it is enough to prove the statement for any $\mathcal{S}^R$ as the limit case of $\mathcal{S}$ then follows by Fatou lemma provided the constant in (4.32) does not depend on R. If $R>0$ is fixed then we use the same approach as in the proof of Proposition 4.4.8, i.e. we will only prove the statement under the additional assumption

$$X_0\in L^p(\Omega\times\mathbb{T}^N\times\mathbb{R})\cap L^\infty(\Omega\times\mathbb{T}^N\times\mathbb{R}).$$

Let X_0^δ be bounded, pathwise smooth and compactly supported regularizations of X_0 such that

$$X_0^\delta\longrightarrow X_0\quad\text{in}\quad L^p(\Omega\times\mathbb{T}^N\times\mathbb{R}),\qquad\|X_0^\delta\|_{L^p_{\omega,x,\xi}}\leq\|X_0\|_{L^p_{\omega,x,\xi}},$$

and $X^\delta=\mathcal{S}^R(t,s)X_0^\delta$ is the unique solution to (4.27). Now, we apply the Itô formula to the function $h(v)=\|v\|_{L^p_{x,\xi}}^p$. If q is the conjugate exponent to p then $h'(v)=p|v|^{p-2}v\in L^q(\mathbb{T}^N\times\mathbb{R})$ and

$$h''(v)=p(p-1)|v|^{p-2}\mathrm{Id}\in\mathscr{L}(L^p(\mathbb{T}^N\times\mathbb{R});L^q(\mathbb{T}^N\times\mathbb{R})).$$

Therefore

$$\begin{aligned}\big\|X^\delta(t)\big\|_{L^p_{x,\xi}}^p&=\big\|X_0^\delta\big\|_{L^p_{x,\xi}}^p\\&\quad-p\int_s^t\int_{\mathbb{T}^N}\int_{\mathbb{R}}\big|X^\delta\big|^{p-2}X^\delta\,a^R(\xi)\cdot\nabla X^\delta\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}r\\&\quad-p\sum_{k=1}^d\int_s^t\int_{\mathbb{T}^N}\int_{\mathbb{R}}\big|X^\delta\big|^{p-2}X^\delta\partial_\xi X^\delta g_k^R(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}\beta_k(r)\\&\quad+\frac{p}{2}\int_s^t\int_{\mathbb{T}^N}\int_{\mathbb{R}}\big|X^\delta\big|^{p-2}X^\delta\,\partial_\xi\big(G^{R,2}\partial_\xi X^\delta\big)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}r\\&\quad+\frac{p(p-1)}{2}\int_s^t\int_{\mathbb{T}^N}\int_{\mathbb{R}}\big|X^\delta\big|^{p-2}\big|\partial_\xi X^\delta\big|^2G^{R,2}(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}r.\end{aligned}$$

Using integration by parts, the second term on the right hand side vanishes. Besides, having known the behavior of X^δ for large ξ, we integrate by parts in the fourth term and obtain the fifth term with opposite sign. To deal with the stochastic term, we also integrate by parts and observe

$$\begin{aligned}&-p\int_{\mathbb{R}}\big|X^\delta\big|^{p-2}X^\delta\partial_\xi X^\delta g_k^R(x,\xi)\,\mathrm{d}\xi\\&\qquad=p(p-1)\int_{\mathbb{R}}\big|X^\delta\big|^{p-2}\partial_\xi X^\delta X^\delta g_k^R(x,\xi)\,\mathrm{d}\xi+p\int_{\mathbb{R}}\big|X^\delta\big|^p\partial_\xi g_k^R(x,\xi)\,\mathrm{d}\xi\end{aligned}$$

hence

$$-p\int_{\mathbb{R}}\big|X^\delta\big|^{p-2}X^\delta\partial_\xi X^\delta g_k^R(x,\xi)\,\mathrm{d}\xi=\int_{\mathbb{R}}\big|X^\delta\big|^p\partial_\xi g_k^R(x,\xi)\,\mathrm{d}\xi$$

and we arrive at

$$\big\|X^\delta(t)\big\|_{L^p_{x,\xi}}^p=\big\|X_0^\delta\big\|_{L^p_{x,\xi}}^p+\sum_{k=1}^d\int_s^t\int_{\mathbb{T}^N}\int_{\mathbb{R}}\big|X^\delta\big|^p\partial_\xi g_k^R(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}\beta_k(r),$$

where the stochastic integral on the right hand side is a martingale with zero expected value. Taking the expectation now yields

$$\mathbb{E}\big\|X^\delta(t)\big\|_{L^p_{x,\xi}}^p=\mathbb{E}\big\|X_0^\delta\big\|_{L^p_{x,\xi}}^p.$$

In order to derive (4.32), we employ the Burkholder-Davis-Gundy inequality and boundedness of $\partial_\xi g_k$:

$$\begin{aligned}
\mathbb{E}\sup_{s\le t\le T}\big\|X^\delta(t)\big\|^p_{L^p_{x,\xi}} &\le \mathbb{E}\big\|X^\delta_0\big\|^p_{L^p_{x,\xi}}\\
&\quad+\sum_{k=1}^d \mathbb{E}\sup_{s\le t\le T}\int_s^t\int_{\mathbb{T}^N}\int_{\mathbb{R}}\big|X^\delta\big|^p\partial_\xi g_k^R(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}\beta_k(r)\\
&\le \mathbb{E}\big\|X^\delta_0\big\|^p_{L^p_{x,\xi}}+C\,\mathbb{E}\bigg(\int_s^T\big\|X^\delta(r)\big\|^{2p}_{L^p_{x,\xi}}\,\mathrm{d}r\bigg)^{\frac12}\\
&\le \mathbb{E}\big\|X^\delta_0\big\|^p_{L^p_{x,\xi}}+\frac12\,\mathbb{E}\sup_{s\le t\le T}\big\|X^\delta(t)\big\|^p_{L^p_{x,\xi}}+C\int_s^T\mathbb{E}\big\|X^\delta(r)\big\|^p_{L^p_{x,\xi}}\,\mathrm{d}r
\end{aligned}$$

hence

$$\mathbb{E}\sup_{s\le t\le T}\big\|X^\delta(t)\big\|^p_{L^p_{x,\xi}}\le C\,\mathbb{E}\big\|X^\delta_0\big\|^p_{L^p_{x,\xi}}.$$

Note, that the constant C does not depend on δ, s, R. Therefore, the fact that the operator norm is equal to 1 as well as the validity of (4.32) follow easily by the same reasoning as in the proof of Proposition 4.4.8. $\square$

Proposition 4.4.15. *Assume that $w\in L^p(\Omega\times\mathbb{T}^N)$ for all $p\in[1,\infty)$. Then for all $n\in[0,\infty)$ there exists $r\in[1,\infty)$ such that*

$$\sup_{0\le s\le T}\mathbb{E}\sup_{s\le t\le T}\big\|\big(\mathcal{S}(t,s)\chi_w\big)(1+|\xi|)^n\big\|_{L^1_{x,\xi}}\le C\Big(1+\|w\|^r_{L^r_{\omega,x}}\Big),$$

where the constant C does not depend on w.

Proof. We will prove that the claim holds true for all $\mathcal{S}^R$ with a constant independent of R. Let us denote by $\psi^{R,x}$ the vector of all x^i-coordinates of the stochastic flow ψ^R, i.e. $\psi^{R,x}_{s,t}(x,\xi)=\big(\psi^{R,1}_{s,t}(x,\xi),\dots,\psi^{R,N}_{s,t}(x,\xi)\big)$. Since it holds, for any $m\in[0,\infty)$,

$$|\chi_w|\le\frac{(1+|w|^2)^m}{(1+|\xi|^2)^m}\mathbf{1}_{|\xi|<|w|}$$

we can estimate

$$\begin{aligned}
\big|\mathcal{S}^R(t,s)\chi_w\big|(1+|\xi|^n)&=\big|\chi_{w(\psi^{R,x}_{s,t}(x,\xi))}(\psi^{R,0}_{s,t}(x,\xi))\big|(1+|\xi|)^n\\
&\le\frac{\big(1+\big|w(\psi^{R,x}_{s,t}(x,\xi))\big|^2\big)^m}{(1+|\psi^{R,0}_{s,t}(x,\xi)|^2)^m}\mathbf{1}_{|\psi^{R,0}_{s,t}(x,\xi)|<|w(\psi^{R,x}_{s,t}(x,\xi))|}(1+|\xi|)^n\\
&\le\frac{(1+|\xi|^2)^{n/2}}{(1+|\psi^{R,0}_{s,t}(x,\xi)|^2)^m}\,\mathcal{S}^R(t,s)\Big[(1+|w|^2)^m\mathbf{1}_{|\xi|<|w|}\Big],
\end{aligned}\tag{4.33}$$

where the exact value of the exponent m will be determined later on. Now, we make use of the classical moment estimate for SDEs that in our setting reads

$$\sup_{\substack{0\le s\le T\\(y,\zeta)\in\mathbb{T}^N\times\mathbb{R}}}\mathbb{E}\sup_{s\le t\le T}\frac{(1+|\varphi^{R,0}_{s,t}(y,\zeta)|^2)^p}{(1+|\zeta|^2)^p}\le C,\qquad\forall p\in[1,\infty),$$

and rewritten in terms of the inverse flow by setting $(x,\xi)=\varphi^R_{s,t}(y,\zeta)$

$$\sup_{\substack{0\le s\le T\\(x,\xi)\in\mathbb{T}^N\times\mathbb{R}}}\mathbb{E}\sup_{s\le t\le T}\frac{(1+|\xi|^2)^p}{(1+|\psi^{R,0}_{s,t}(x,\xi)|^2)^p}\le C,\qquad\forall p\in[1,\infty),\tag{4.34}$$

with a constant independent of R. Therefore, employing (4.33), the Young inequality, (4.34) and Proposition 4.4.14 we obtain by a suitable choice of m

$$\begin{aligned}&\sup_{0\le s\le T}\mathbb{E}\sup_{s\le t\le T}\int_{\mathbb{T}^N}\int_{\mathbb{R}}\big|\mathcal{S}^R(t,s)\chi_w\big|(1+|\xi|)^n\,\mathrm{d}\xi\,\mathrm{d}x\\&\quad\le C\sup_{0\le s\le T}\mathbb{E}\sup_{s\le t\le T}\int_{\mathbb{T}^N}\int_{\mathbb{R}}\frac{(1+|\xi|^2)^n}{(1+|\psi^{R,0}_{s,t}(x,\xi)|^2)^{2m}}\mathrm{d}\xi\,\mathrm{d}x\\&\qquad+C\sup_{0\le s\le T}\mathbb{E}\sup_{s\le t\le T}\int_{\mathbb{T}^N}\int_{\mathbb{R}}\Big|\mathcal{S}^R(t,s)\Big[(1+|w|^2)^m\mathbf{1}_{|\xi|<|w|}\Big]\Big|^2\mathrm{d}\xi\,\mathrm{d}x\\&\quad\le C+C\big\|(1+|w|^2)^m\mathbf{1}_{|\xi|<|w|}\big\|^2_{L^2_{\omega,x,\xi}}\le C\Big(1+\|w\|^{4m+1}_{L^{4m+1}_{\omega,x}}\Big)\end{aligned}$$

which completes the proof. □

4.5 Convergence of the BGK approximation

In this final section, we investigate the limit of the stochastic BGK model as $\varepsilon\to0$ and prove our main result, Theorem 4.2.1. To be more precise, we consider the following weak formulation of (4.3), which is satisfied by F^ε, and show its convergence to the kinetic formulation of (4.1). Let $\varphi\in C_c^\infty([0,T)\times\mathbb{T}^N\times\mathbb{R})$ then

$$\begin{aligned}&\int_0^T\big\langle F^\varepsilon(t),\partial_t\varphi(t)\big\rangle\,\mathrm{d}t+\big\langle F^\varepsilon_0,\varphi(0)\big\rangle+\int_0^T\big\langle F^\varepsilon(t),a\cdot\nabla\varphi(t)\big\rangle\,\mathrm{d}t\\&=-\frac{1}{\varepsilon}\int_0^T\big\langle\mathbf{1}_{u^\varepsilon(t)>\xi}-F^\varepsilon(t),\varphi(t)\big\rangle\,\mathrm{d}t+\int_0^T\big\langle\partial_\xi F^\varepsilon(t)\,\varPhi\,\mathrm{d}W(t),\varphi(t)\big\rangle\\&\quad+\frac12\int_0^T\big\langle G^2\partial_\xi F^\varepsilon(t),\partial_\xi\varphi(t)\big\rangle\,\mathrm{d}t.\end{aligned}\tag{4.35}$$

A similar expression holds true also for h^ε, namely, it satisfies the weak formulation of (4.30). However, as in the following we restrict our attention to the representatives $F^\varepsilon(t)$ and $h^\varepsilon(t)$, respectively, given by (4.15) and (4.31), respectively, we point out that both are true even in a stronger sense. For the case of $h^\varepsilon(t)$, we have: let $\varphi\in C_c^\infty(\mathbb{T}^N\times\mathbb{R})$ then it holds for all $t\in[0,T]$

$$\begin{aligned}\big\langle h^\varepsilon(t),\varphi\big\rangle&=\big\langle h^\varepsilon_0,\varphi\big\rangle+\int_0^t\big\langle h^\varepsilon(s),a\cdot\nabla\varphi\big\rangle\,\mathrm{d}s\\&\quad+\frac1\varepsilon\int_0^t\big\langle\mathbf{1}_{u^\varepsilon(s)>\xi}-\mathcal{S}(s,0)\mathbf{1}_{0>\xi}-h^\varepsilon(s),\varphi\big\rangle\,\mathrm{d}s\\&\quad-\int_0^t\big\langle\partial_\xi h^\varepsilon(s)\,\varPhi\,\mathrm{d}W(s),\varphi\big\rangle-\frac12\int_0^t\big\langle G^2\partial_\xi h^\varepsilon(s),\partial_\xi\varphi\big\rangle\,\mathrm{d}s.\end{aligned}\tag{4.36}$$

Proof of Theorem 4.2.1. Taking the limit in (4.35) is quite straightforward in all the terms apart from the first one on the right hand side and can be done immediately. Remark, that according to the representation formula (4.15) it holds that the set of solutions $\{F^\varepsilon;\, \varepsilon\in(0,1)\}$ is bounded in $L^\infty_{\mathcal{P}}(\Omega\times[0,T]\times\mathbb{T}^N\times\mathbb{R})$, more precisely, $F^\varepsilon\in[0,1]$, $\varepsilon\in(0,1)$. Therefore, by the Banach-Alaoglu theorem, there exists $F\in L^\infty_{\mathcal{P}}(\Omega\times[0,T]\times\mathbb{T}^N\times\mathbb{R})$ such that, up to subsequences,

$$F^\varepsilon \xrightarrow{w^*} F \quad \text{in} \quad L^\infty_{\mathcal{P}}(\Omega\times[0,T]\times\mathbb{T}^N\times\mathbb{R}). \tag{4.37}$$

Hence, almost surely,

$$\begin{aligned}
\int_0^T \big\langle F^\varepsilon(t),\, \partial_t\varphi(t)\big\rangle\,\mathrm{d}t &\longrightarrow \int_0^T \big\langle F(t),\, \partial_t\varphi(t)\big\rangle\,\mathrm{d}t,\\
\int_0^T \big\langle F^\varepsilon(t), a\cdot\nabla\varphi(t)\big\rangle\,\mathrm{d}t &\longrightarrow \int_0^T \big\langle F(t), a\cdot\nabla\varphi(t)\big\rangle\,\mathrm{d}t,\\
\frac12\int_0^T \big\langle G^2\partial_\xi F^\varepsilon(t), \partial_\xi\varphi(t)\big\rangle\,\mathrm{d}t &\longrightarrow \frac12\int_0^T \big\langle G^2\partial_\xi F(t), \partial_\xi\varphi(t)\big\rangle\,\mathrm{d}t.
\end{aligned}$$

and, according to the hypotheses on the initial data,

$$\big\langle F_0^\varepsilon, \varphi(0)\big\rangle \longrightarrow \big\langle \mathbf{1}_{u_0>\xi}, \varphi(0)\big\rangle.$$

We intend to prove a similar convergence result for the stochastic term as well. Since

$$\big\langle F^\varepsilon, \partial_\xi(g_k\varphi)\big\rangle \longrightarrow \big\langle F, \partial_\xi(g_k\varphi)\big\rangle, \qquad \text{a.e. } (\omega,t)\in\Omega\times[0,T],$$

and, due to the boundedness of F^ε and the assumptions on g_k,

$$\big|\big\langle F^\varepsilon, \partial_\xi(g_k\varphi)\big\rangle\big| \le C,$$

the dominated convergence theorem for stochastic integrals gives (up to subsequences) the desired almost sure convergence

$$\int_0^T \big\langle \partial_\xi F^\varepsilon(t)\,\varPhi\,\mathrm{d}W(t), \varphi(t)\big\rangle \longrightarrow \int_0^T \big\langle \partial_\xi F(t)\,\varPhi\,\mathrm{d}W(t), \varphi(t)\big\rangle.$$

Furthermore, multiplying (4.35) by ε yields, almost surely,

$$\int_0^T \big\langle \mathbf{1}_{u^\varepsilon(t)>\xi} - F^\varepsilon(t), \varphi(t)\big\rangle\,\mathrm{d}t \longrightarrow 0 \tag{4.38}$$

and, in particular,

$$\partial_\xi \mathbf{1}_{u^\varepsilon>\xi} - \partial_\xi F^\varepsilon \longrightarrow 0 \tag{4.39}$$

in the sense of distributions over $(0,T)\times\mathbb{T}^N\times\mathbb{R}$ almost surely. In order to obtain the convergence in the remaining term of (4.35) and in view of the kinetic formulation of (4.1), we need to show that the term $\frac{1}{\varepsilon}(\mathbf{1}_{u^\varepsilon>\xi}-F^\varepsilon)$ can be written as $\partial_\xi m^\varepsilon$ where m^ε is a random nonnegative measure over $[0,T]\times\mathbb{T}^N\times\mathbb{R}$ bounded uniformly in ε. However, if we define

$$\begin{aligned}
m^\varepsilon(\xi) &= \frac{1}{\varepsilon}\int_{-\infty}^{\xi} \big(\mathbf{1}_{u^\varepsilon>\zeta} - F^\varepsilon(\zeta)\big)\,\mathrm{d}\zeta\\
&= \frac{1}{\varepsilon}\int_{-\infty}^{\xi} \big(\mathbf{1}_{u^\varepsilon>\zeta} - \mathcal{S}(t,0)\mathbf{1}_{0>\zeta} - h^\varepsilon(\zeta)\big)\,\mathrm{d}\zeta,
\end{aligned} \tag{4.40}$$

it is easy to check that $m^\varepsilon \geq 0$ since $F^\varepsilon \in [0,1]$. Indeed, $m^\varepsilon(-\infty) = m^\varepsilon(\infty) = 0$ and $m^\varepsilon(t,x,\cdot)$ is increasing if $\xi \in (-\infty, u^\varepsilon(t,x))$ and decreasing if $\xi \in (u^\varepsilon(t,x), \infty)$.

Due to the convergence in (4.35) it can be seen that for almost every $\omega \in \Omega$ there exists a distribution $m(\omega)$ such that, almost surely,

$$\int_0^T \big\langle m^\varepsilon, \varphi(t)\big\rangle \,\mathrm{d}t \longrightarrow \int_0^T \big\langle m, \varphi(t)\big\rangle \,\mathrm{d}t, \tag{4.41}$$

for any $\varphi \in C_c^\infty([0,T) \times \mathbb{T}^N \times \mathbb{R})$. Besides, the conditions on test functions can be relaxed so that (4.41) holds true for any $\varphi \in C_c^\infty([0,T] \times \mathbb{T}^N \times \mathbb{R})$. Now, it remains to verify that m is a kinetic measure. The following proposition will be useful.

Proposition 4.5.1. *The set of local densities $\{u^\varepsilon;\, \varepsilon \in (0,1)\}$ is bounded in*

$$L^p(\Omega; L^\infty(0,T; L^p(\mathbb{T}^N))), \quad \forall p \in [1,\infty).$$

Proof. We need to find a uniform estimate for u^ε. It follows from the definition of u^ε (4.29) and (4.15) that

$$\begin{aligned} u^\varepsilon(t,x) &= \mathrm{e}^{-\frac{t}{\varepsilon}} \int_\mathbb{R} \big(\mathcal{S}(t,0)\mathbf{1}_{u_0^\varepsilon > \xi} - \mathbf{1}_{0>\xi}\big)\,\mathrm{d}\xi \\ &\quad + \frac{1}{\varepsilon}\int_0^t \mathrm{e}^{-\frac{t-s}{\varepsilon}} \int_\mathbb{R} \big(\mathcal{S}(t,s)\mathbf{1}_{u^\varepsilon(s)>\xi} - \mathbf{1}_{0>\xi}\big)\,\mathrm{d}\xi\,\mathrm{d}s. \end{aligned}$$

Let us now define the following auxiliary function

$$H(s) = \left| \int_\mathbb{R} \big(\mathcal{S}(t,s)\mathbf{1}_{u^\varepsilon(s)>\xi} - \mathbf{1}_{0>\xi}\big)\,\mathrm{d}\xi \right|.$$

Then

$$H(t) \leq \mathrm{e}^{-\frac{t}{\varepsilon}} H(0) + (1 - \mathrm{e}^{-\frac{t}{\varepsilon}}) \max_{0\leq s\leq t} H(s)$$

and we conclude that $H(t) \leq H(0)$, $t \in [0,T]$. In order to estimate $H(0)$, we make use of Proposition 4.4.15 and Corollary 4.4.11. If $p = 1$ they can be used directly

$$\begin{aligned} \mathbb{E} \sup_{0\leq t\leq T} \int_{\mathbb{T}^N} |u^\varepsilon(t,x)|\,\mathrm{d}x &\leq \mathbb{E} \sup_{0\leq t\leq T} \int_{\mathbb{T}^N}\int_\mathbb{R} \big|\mathcal{S}(t,0)\mathbf{1}_{u_0^\varepsilon>\xi} - \mathbf{1}_{0>\xi}\big|\,\mathrm{d}\xi\,\mathrm{d}x \\ &\leq \mathbb{E} \sup_{0\leq t\leq T} \big\|\mathcal{S}(t,0)\chi_{u_0^\varepsilon}\big\|_{L^1_{x,\xi}} + \mathbb{E} \sup_{0\leq t\leq T} \big\|\mathcal{S}(t,0)\mathbf{1}_{0>\xi} - \mathbf{1}_{0>\xi}\big\|_{L^1_{x,\xi}} \\ &\leq C\Big(1 + \|u_0^\varepsilon\|_{L^{r_1}_{\omega,x}}^{r_1}\Big), \end{aligned}$$

whereas the case of $p \in (1,\infty)$ can be dealt with by the Hölder inequality and the fact that

$$\big|\mathcal{S}(t,0)\mathbf{1}_{u_0^\varepsilon>\xi} - \mathbf{1}_{0>\xi}\big|^p = \big|\mathcal{S}(t,0)\mathbf{1}_{u_0^\varepsilon>\xi} - \mathbf{1}_{0>\xi}\big|.$$

Indeed,

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\int_{\mathbb{T}^N}|u^\varepsilon(t,x)|^p\,\mathrm{d}x &\le \mathbb{E}\sup_{0\le t\le T}\int_{\mathbb{T}^N}\Big(\int_{\mathbb{R}}\big|\mathcal{S}(t,0)\mathbf{1}_{u_0^\varepsilon>\xi}-\mathbf{1}_{0>\xi}\big|\,\mathrm{d}\xi\Big)^p\mathrm{d}x\\ &\le C\,\mathbb{E}\sup_{0\le t\le T}\big\|\mathcal{S}(t,0)\chi_{u_0^\varepsilon}(1+|\xi|)^p\big\|_{L^1_{x,\xi}}\\ &\quad+C\,\mathbb{E}\sup_{0\le t\le T}\big\|\big(\mathcal{S}(t,0)\mathbf{1}_{0>\xi}-\mathbf{1}_{0>\xi}\big)(1+|\xi|)^p\big\|_{L^1_{x,\xi}}\\ &\le C\Big(1+\|u_0^\varepsilon\|^{r_p}_{L^{r_p}_{\omega,x}}\Big).\end{aligned}$$

The above exponents r_p are given by Proposition 4.4.15 and the proof is complete. ◻

Corollary 4.5.2. *For any $n\in[0,\infty)$ it holds*

$$\sup_{0\le t\le T}\mathbb{E}\big\|h^\varepsilon(t)(1+|\xi|)^n\big\|_{L^1_{x,\xi}}\le C.$$

Proof. It follows from (4.31), Proposition 4.4.15, Corollary 4.4.11 and Proposition 4.5.1 that

$$\begin{aligned}\sup_{0\le t\le T}\mathbb{E}\big\|h^\varepsilon(t)(1+|\xi|)^n\big\|_{L^1_{x,\xi}} &\le \sup_{0\le s\le t\le T}\mathbb{E}\big\|\mathcal{S}(t,s)\chi_{u^\varepsilon(s)}(1+|\xi|)^n\big\|_{L^1_{x,\xi}}\\ &\quad+\sup_{0\le s\le t\le T}\mathbb{E}\big\|\big(\mathbf{1}_{0>\xi}-\mathcal{S}(s,0)\mathbf{1}_{0>\xi}\big)(1+|\xi|)^n\big\|_{L^1_{x,\xi}}\\ &\le C\Big(1+\sup_{0\le s\le T}\|u^\varepsilon(s)\|^r_{L^r_{\omega,x}}\Big)\le C.\end{aligned}$$

◻

As a consequence, the assumptions of [16, Theorem 5] are satisfied for $\nu^\varepsilon_{t,x}=\delta_{u^\varepsilon(t,x)=\xi}$ and hence there exists a kinetic measure $\nu_{t,x}$ vanishing at infinity such that $\nu^\varepsilon\to\nu$ in the sense given by this theorem. We deduce from (4.39) that $\partial_\xi F=-\nu$ hence F is a kinetic function in the sense of [16, Definition 4].

Remark, that it follows now from (4.40) that the function $m^\varepsilon(t)$ satisfies

$$\sup_{0\le t\le T}\mathbb{E}\big\|m^\varepsilon(t)(1+|\xi|)^n\big\|_{L^1_{x,\xi}}\le C(\varepsilon),$$

for any ε fixed. Nevertheless, we do not know yet if this fact holds true also uniformly in ε. Towards this end, we will study the weak formulation for h^ε and employ a suitable test function.

Proposition 4.5.3. *For any $p\in[0,\infty)$ it holds*

$$\mathbb{E}\int_{[0,T]\times\mathbb{T}^N\times\mathbb{R}}|\xi|^{2p}\,\mathrm{d}m^\varepsilon(t,x,\xi)\le C. \tag{4.42}$$

Proof. Let $p\in[1/2,\infty)$. Regarding (4.36), we need to test by $\varphi(\xi)=\frac{\xi^{2p+1}}{2p+1}$. Due to the behavior of m^ε and h^ε for large ξ we can consider test functions which are not compactly supported in ξ, however, in this case the stochastic integral is not necessarily a martingale. Therefore we will first employ the truncation $\varphi^\delta(\xi)=\varphi(\xi)k_\delta(\xi)$ and then pass to the limit. We have

$$0 \le \mathbb{E} \int_0^T \left\langle m^\varepsilon(t), \partial_\xi \varphi^\delta \right\rangle \mathrm{d}t = \mathbb{E} \left\langle h_0^\varepsilon, \varphi^\delta \right\rangle - \mathbb{E} \left\langle h^\varepsilon(T), \varphi^\delta \right\rangle - \frac{1}{2} \mathbb{E} \int_0^T \left\langle G^2 \partial_\xi h^\varepsilon(t), \partial_\xi \varphi^\delta \right\rangle \mathrm{d}t.$$

The first and the second term on the right hand side can be estimated by Corollary 4.5.2

$$\mathbb{E} \left\langle h_0^\varepsilon, \varphi^\delta \right\rangle - \mathbb{E} \left\langle h^\varepsilon(T), \varphi^\delta \right\rangle \le C,$$

while for the remaining term we first employ the growth properties of G^2 and $\partial_\xi G^2$ to obtain

$$\begin{aligned} &\mathbb{E} \int_0^T \left\langle G^2 \partial_\xi h^\varepsilon(t), \partial_\xi \varphi^\delta \right\rangle \mathrm{d}t \\ &\qquad \le C\, \mathbb{E} \int_0^T \left\langle |h^\varepsilon(t)|, (1+|\xi|) \partial_\xi \varphi^\delta + (1+|\xi|^2) \partial_\xi^2 \varphi^\delta \right\rangle \mathrm{d}t \\ &\qquad \le C\, \mathbb{E} \int_0^T \left\langle |h^\varepsilon(t)|, (1+|\xi|)^{2p+3} \right\rangle \mathrm{d}t \le C. \end{aligned}$$

The constant C is independent of δ thus the claim follows.

If $p=0$ a suitable modification in the above estimation leads to the proof in this case whereas the case of $p \in (0,1/2)$ follows from (4.42) for $p=0$ and $p=1/2$ due to the fact that $|\xi|^{2p} \le 1 + |\xi|$. □

Setting $p=0$ in (4.42) we regard m^ε as random variables with values in $\mathcal{M}_b([0,T] \times \mathbb{T}^N \times \mathbb{R})$, the space of bounded Borel measures on $[0,T] \times \mathbb{T}^N \times \mathbb{R}$ whose norm is given by the total variation of measures. We deduce that the set of laws $\{\mathbb{P} \circ [m^\varepsilon]^{-1};\, \varepsilon \in (0,1)\}$ is tight and therefore any sequence has a weakly convergent subsequence due to the Prokhorov theorem. Consequently, the law of m is supported in $\mathcal{M}_b([0,T] \times \mathbb{T}^N \times \mathbb{R})$. Besides, m is nonnegative as it holds true for all m^ε. Moreover, since $C_0([0,T] \times \mathbb{T}^N \times \mathbb{R})$, the space of continuous functions vanishing at infinity equipped with the supremum norm, is the predual of $\mathcal{M}_b([0,T] \times \mathbb{T}^N \times \mathbb{R})$ and $C_c^\infty([0,T] \times \mathbb{T}^N \times \mathbb{R})$ is dense in $C_0([0,T] \times \mathbb{T}^N \times \mathbb{R})$ it can be seen that (4.41) holds true for any $\varphi \in C_0([0,T] \times \mathbb{T}^N \times \mathbb{R})$. Now, it is left to verify the three points of the definition of a kinetic measure [16, Definition 1]. The second requirement giving the behavior for large ξ follows from the above uniform estimate (4.42). Indeed, let (k_δ) be a truncation on $\mathbb{R}$, e.g. the set of functions defined in the proof of Proposition 4.4.8, then

$$\begin{aligned} \mathbb{E} \int_{[0,T] \times \mathbb{T}^N \times \mathbb{R}} |\xi|^{2p} \,\mathrm{d}m(t,x,\xi) &\le \liminf_{\delta \to 0} \mathbb{E} \int_{[0,T] \times \mathbb{T}^N \times \mathbb{R}} |\xi|^{2p} k_\delta(\xi) \,\mathrm{d}m(t,x,\xi) \\ &= \liminf_{\delta \to 0} \lim_{\varepsilon \to 0} \mathbb{E} \int_{[0,T] \times \mathbb{T}^N \times \mathbb{R}} |\xi|^{2p} k_\delta(\xi) \,\mathrm{d}m^\varepsilon(t,x,\xi) \le C. \end{aligned}$$

As a consequence, m vanishes for large ξ. The first point of [16, Definition 1] is straightforward for $\phi \in C_0([0,T] \times \mathbb{T}^N \times \mathbb{R})$ as a pointwise limit of measurable functions is measurable. The case of $\phi \in C_b([0,T] \times \mathbb{T}^N \times \mathbb{R})$ now follows by employing the truncation (k_δ) together with the dominated convergence theorem as $\delta \to 0$ and the behavior of m at for large ξ. In order to show predictability of the process

$$t \longmapsto \int_{[0,t]\times\mathbb{T}^N\times\mathbb{R}} \phi(x,\xi)\,\mathrm{d}m(s,x,\xi)$$

in the case of $\phi \in C_0(\mathbb{T}^N \times \mathbb{R})$ let us remark that due to (4.36) it is the pointwise limit (in ω and t) of predictable processes

$$t \longmapsto \int_{[0,t]\times\mathbb{T}^N\times\mathbb{R}} \phi(x,\xi)\,\mathrm{d}m^\varepsilon(s,x,\xi)$$

and hence is also measurable with respect to the predictable σ-algebra. The case of $\phi \in C_b(\mathbb{T}^N \times \mathbb{R})$ can be verified by using truncations as above. Therefore, we have proved that m is a kinetic measure.

Finally, we deduce that F satisfies the generalized kinetic formulation (4.8) and thus is a generalized kinetic solution to (4.1). Since any generalized kinetic solution is actually a kinetic one, due to the reduction theorem [16, Theorem 11], it follows that $F = \mathbf{1}_{u>\xi}$ and $\nu = \delta_u$, where $u \in L^p(\Omega \times [0,T] \times \mathbb{T}^N)$ is the unique kinetic solution to (4.1). Therefore, it only remains to verify the strong convergence of f^ε and u^ε to χ_u and u, respectively.

According to (4.37), we deduce for $f^\varepsilon = F^\varepsilon - \mathbf{1}_{0>\xi}$ that

$$f^\varepsilon \overset{w^*}{\longrightarrow} \chi_u \qquad \text{in} \qquad L^\infty(\Omega \times [0,T] \times \mathbb{T}^N \times \mathbb{R}),$$

and by (4.38) it holds

$$\chi_{u^\varepsilon} \longrightarrow \chi_u \qquad \text{in} \qquad \mathcal{D}'((0,T) \times \mathbb{T}^N \times \mathbb{R}),\ \mathbb{P}\text{-a.s.}.$$

Besides, $\{\chi_{u^\varepsilon};\, \varepsilon \in (0,1)\}$ is bounded in $L^\infty(\Omega \times [0,T] \times \mathbb{T}^N \times \mathbb{R})$ hence (up to subsequences) it converges weak* in this space and since $C_c^\infty((0,T) \times \mathbb{T}^N \times \mathbb{R})$ is separable and dense in $L^1([0,T] \times \mathbb{T}^N \times \mathbb{R})$, it follows that χ_u is the limit, i.e.

$$\chi_{u^\varepsilon} \overset{w^*}{\longrightarrow} \chi_u \qquad \text{in} \qquad L^\infty(\Omega \times [0,T] \times \mathbb{T}^N \times \mathbb{R}).$$

Furthermore, according to Proposition 4.5.1, it holds for any $n \in [0,\infty)$

$$\sup_{0\leq t\leq T} \mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}} \big(|\chi_{u^\varepsilon(t)}| + |\chi_{u(t)}|\big)(1+|\xi|)^n \,\mathrm{d}\xi\,\mathrm{d}x \leq C, \tag{4.43}$$

hence we can relax the conditions on test functions and obtain the strong convergence $\chi_{u^\varepsilon} \to \chi_u$ in $L^2(\Omega \times [0,T] \times \mathbb{T}^N \times \mathbb{R})$. Indeed,

$$\begin{aligned} &\mathbb{E} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\chi_{u^\varepsilon} - \chi_u|^2 \,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad = \mathbb{E} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\chi_{u^\varepsilon}| - 2\chi_{u^\varepsilon}\chi_u + |\chi_u| \,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t \longrightarrow 0 \end{aligned} \tag{4.44}$$

since for the first term on the right hand side we have

$$\mathbb{E} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\chi_{u^\varepsilon}| \,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t = \mathbb{E} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} \big(\chi_{u^\varepsilon}\mathbf{1}_{\xi>0} - \chi_{u^\varepsilon}\mathbf{1}_{\xi<0}\big) \,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$

where $\mathbf{1}_{\xi>0}, \mathbf{1}_{\xi<0}$ can be taken as test functions due to (4.43) and for the second term on the right hand side we consider χ_u as a test function. As $|\chi_\alpha - \chi_\beta|^p = |\chi_\alpha - \chi_\beta|$ we conclude also the strong convergence in all $L^p(\Omega \times [0,T] \times \mathbb{T}^N \times \mathbb{R})$, $p \in [1,\infty)$.

Moreover, a similar approach can be used to prove the convergence of f^ε. Indeed, the same calculation as in (4.44) gives

$$f^\varepsilon \longrightarrow \chi_u \qquad \text{in} \qquad L^2(\Omega \times [0,T] \times \mathbb{T}^N \times \mathbb{R})$$

and using the uniform bound of $\{f^\varepsilon;\, \varepsilon \in (0,1)\}$ in $L^\infty(\Omega \times [0,T] \times \mathbb{T}^N \times \mathbb{R})$ we deduce the convergence in $L^p(\Omega \times [0,T] \times \mathbb{T}^N \times \mathbb{R})$ for all $p \in [1,\infty)$.

Eventually, by the properties of the equilibrium function we have

$$u^\varepsilon \longrightarrow u \qquad \text{in} \qquad L^1(\Omega \times [0,T] \times \mathbb{T}^N).$$

On the other hand, it follows from Proposition 4.5.1 that the set $\{u^\varepsilon;\, \varepsilon \in (0,1)\}$ is bounded in $L^p(\Omega \times [0,T] \times \mathbb{T}^N)$, for all $p \in [1,\infty)$, hence by application of the Hölder inequality, we get also the strong convergence

$$u^\varepsilon \longrightarrow u \qquad \text{in} \qquad L^p(\Omega \times [0,T] \times \mathbb{T}^N) \qquad \forall p \in [1,\infty).$$

Therefore, the proof of convergence in the stochastic BGK model is complete. □

Chapter 5

On Weak Solutions of Stochastic Differential Equations

Abstract: A new proof of existence of weak solutions to stochastic differential equations with continuous coefficients based on ideas from infinite-dimensional stochastic analysis is presented. The proof is fairly elementary, in particular, neither theorems on representation of martingales by stochastic integrals nor results on almost sure representation for tight sequences of random variables are needed. In the second part we show that the same method may be used even if the linear growth hypothesis is replaced with a suitable Lyapunov condition.

Results of this chapter were published under the titles:

- M. Hofmanová, J. Seidler, *On Weak Solutions of Stochastic Differential Equations*, Stoch. Anal. Appl. **30** (1) (2012) 100–121,

- M. Hofmanová, J. Seidler, *On Weak Solutions of Stochastic Differential Equations II.*, Stoch. Anal. Appl., to appear.

5.1 Introduction

In this paper, we provide a modified proof of Skorokhod's classical theorem on existence of (weak) solutions to a stochastic differential equation

$$\mathrm{d}X = b(t,X)\,\mathrm{d}t + \sigma(t,X)\,\mathrm{d}W, \quad X(0) = \varphi,$$

where $b : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ and $\sigma : [0,T] \times \mathbb{R}^m \to \mathbb{M}_{m\times n}$ are Borel functions of at most linear growth and continuous in the second variable. (Henceforward, by $\mathbb{M}_{m\times n}$ we shall denote the space of all m-by-n matrices over $\mathbb{R}$ endowed with the Hilbert-Schmidt norm $\|A\| = (\mathrm{Tr}\, AA^*)^{1/2}$.) Our proof combines tools that were proposed for handling weak solutions of stochastic evolution equations in infinite-dimensional spaces, where traditional methods cease to work, with results on preservation of the local martingale property under convergence in law. In finite-dimensional situation, the "infinite-dimensional" methods simplify considerably and in our opinion the alternative proof based on them is more lucid and elementary than the standard one. A positive teaching experience of the second author was, in fact, the main motivation for writing this paper. Moreover, we believe that the reader may find the comparison with other available approaches illuminating.

To explain our argument more precisely, let us recall the structure of the usual proof; for notational simplicity, we shall consider (in the informal introduction only) autonomous equations. Kiyosi Itô showed in his seminal papers (see e.g. [39], [40]) that a stochastic differential equation

$$\mathrm{d}X = b(X)\,\mathrm{d}t + \sigma(X)\,\mathrm{d}W \tag{5.1}$$

$$X(0) = \varphi \tag{5.2}$$

driven by an n-dimensional Wiener process W has a unique solution provided that $b : \mathbb{R}^m \to \mathbb{R}^m$, $\sigma : \mathbb{R}^m \to \mathbb{M}_{m\times n}$ are Lipschitz continuous functions. A next important step was taken by A. Skorokhod ([71], [72]) in 1961, who proved that there exists a solution to (5.1), (5.2) if b and σ are continuous functions of at most linear growth, i.e.

$$\sup_{x\in\mathbb{R}^m} \frac{\|b(x)\| + \|\sigma(x)\|}{1 + \|x\|} < \infty.$$

It was realized only later that two different concepts of a solution are involved: for Lipschitzian coefficients, there exists an $(\mathscr{F}_t)$-progressively measurable process in $\mathbb{R}^m$ solving (5.1) and such that $X(0) = \varphi$, whenever $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ is a stochastic basis carrying an n-dimensional $(\mathscr{F}_t)$-Wiener process and φ is an $\mathscr{F}_0$-measurable function. (We say that (5.1), (5.2) has a strong solution.) On the other hand, for continuous coefficients, a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$, an n-dimensional $(\mathscr{F}_t)$-Wiener process W and an $(\mathscr{F}_t)$-progressively measurable process X may be found such that X solves (5.1) and $X(0)$ and φ have the same law. (We speak about existence of a weak solution to (5.1), (5.2) in such a case.) It is well known that this difference is substantial in general: under assumptions of the Skorokhod theorem strong solutions need not exist (see [5]).

Skorokhod's existence theorem is remarkable not only by itself, but also because of the method of its proof. To present it, we need some notation: if M and N are continuous real local martingales, then by $\langle M\rangle$ we denote the quadratic variation of M and by $\langle M, N\rangle$ the cross-variation of M and N. Let $M = (M^i)_{i=1}^m$ and $N = (N^j)_{j=1}^n$ be continuous local martingales with values in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. By $\langle\!\langle M\rangle\!\rangle$ we denote the tensor quadratic variation of M, $\langle\!\langle M\rangle\!\rangle = (\langle M^i, M^k\rangle)_{i,k=1}^m$, and we set $\langle M\rangle = \mathrm{Tr}\langle\!\langle M\rangle\!\rangle$.

Analogously, we define

$$M \otimes N = \left(M^i N^j\right)_{i=1}^{m}{}_{j=1}^{n}, \quad \langle\!\langle M, N \rangle\!\rangle = \left(\langle M^i, N^j \rangle\right)_{i=1}^{m}{}_{j=1}^{n}.$$

Let X and Y be random variables with values in the same measurable space $(E, \mathscr{E})$, we write $X \overset{d}{\sim} Y$ if X and Y have the same law on $\mathscr{E}$. Similarly, $X \overset{d}{\sim} \nu$ means that the law of X is a probability measure ν on $\mathscr{E}$.

Let

$$\mathrm{d}X_r = b_r(X_r)\,\mathrm{d}t + \sigma_r(X_r)\,\mathrm{d}W, \quad X_r(0) = \varphi$$

be a sequence of equations which have strong solutions and approximate (5.1) in a suitable sense. (We shall approximate b and σ by Lipschitz continuous functions having the same growth as b and σ, but likewise it is possible to use e.g. finite difference approximations.) The linear growth hypothesis makes it possible to prove that

$$\textit{the laws of } \{X_r;\ r \geq 1\} \textit{ are tight,} \tag{5.3}$$

that is, form a relatively weakly compact set of measures on the space of continuous trajectories. Then Skorokhod's theorem on almost surely converging realizations of converging laws (see e.g. [18], Theorem 11.7.2) may be invoked, which yields a subsequence $\{X_{r_k}\}$ of $\{X_r\}$, a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ and sequences $\{\tilde{X}_k;\ k \geq 0\}$, $\{\tilde{W}_k;\ k \geq 0\}$ such that

$$(X_{r_k}, W) \overset{d}{\sim} (\tilde{X}_k, \tilde{W}_k),\ \ k \geq 1; \quad (\tilde{X}_k, \tilde{W}_k) \longrightarrow (\tilde{X}_0, \tilde{W}_0),\ \tilde{\mathbb{P}}\text{-a.s..} \tag{5.4}$$

It is claimed that $\tilde{X}_0$ is the (weak) solution looked for. Skorokhod's papers [71] and [72] are written in a very concise way and details of proofs are not offered; nowadays standard version of Skorokhod's proof is as follows (see [73], Theorem 6.1.6, [37], Theorem IV.2.2, [42], Theorem 5.4.22): under a suitable integrability assumption upon the initial condition,

$$M_k = X_{r_k} - X_{r_k}(0) - \int_0^{\cdot} b_{r_k}(X_{r_k}(s))\,\mathrm{d}s$$

is a martingale with a (tensor) quadratic variation

$$\langle\!\langle M_k \rangle\!\rangle = \int_0^{\cdot} \sigma_{r_k}(X_{r_k}(s))\sigma_{r_k}^*(X_{r_k}(s))\,\mathrm{d}s,$$

for all $k \geq 1$. Equality in law (5.4) implies that also

$$\tilde{M}_k = \tilde{X}_k - \tilde{X}_k(0) - \int_0^{\cdot} b_{r_k}(\tilde{X}_k(s))\,\mathrm{d}s$$

are martingales for $k \geq 1$, with quadratic variations

$$\langle\!\langle \tilde{M}_k \rangle\!\rangle = \int_0^{\cdot} \sigma_{r_k}(\tilde{X}_k(s))\sigma_{r_k}^*(\tilde{X}_k(s))\,\mathrm{d}s.$$

Using convergence $\tilde{\mathbb{P}}$-almost everywhere, it is possible to show that

$$\tilde{M}_0 = \tilde{X}_0 - \tilde{X}_0(0) - \int_0^{\cdot} b(\tilde{X}_0(s))\,\mathrm{d}s$$

is a martingale with a quadratic variation

$$\langle\langle\tilde{M}_0\rangle\rangle = \int_0^{\cdot} \sigma(\tilde{X}_0(s))\sigma^*(\tilde{X}_0(s))\,\mathrm{d}s.$$

By the integral representation theorem for martingales with an absolutely continuous quadratic variation (see e.g. [42], Theorem 3.4.2, or [8], Theorem II.7.1'), there exists a Wiener process $\hat{W}$ (on an extended probability space) satisfying

$$\tilde{M}_0 = \int_0^{\cdot} \sigma(\tilde{X}_0(s))\,\mathrm{d}\hat{W}(s).$$

Therefore, $(\hat{W}, \tilde{X}_0)$ is a weak solution to (5.1), (5.2). (In the cited books, martingale problems are used instead of weak solutions. Then the integral representation theorem is hidden in the construction of a weak solution from a solution to the martingale problem, so a complete proof is essentially the one sketched above.)

This procedure has two rather nontrivial inputs: the Skorokhod representation theorem, and the integral representation theorem whose proof, albeit based on a simple and beautiful idea, becomes quite technical if the space dimension is greater than one. An alternative approach to identification of the limit was discovered recently (see [11], [60]) in the course of study of stochastic wave maps between manifolds, where integral representation theorems for martingales are no longer available. The new method, which refers only to basic properties of martingales and stochastic integrals, may be described in the case of the problem (5.1), (5.2) in the following way: One starts again with a sequence $\{(\tilde{X}_k, \tilde{W}_k)\}$ such that (5.4) holds true. If the initial condition is p-integrable for some $p > 2$, it can be shown in a straightforward manner, using the almost sure convergence, that

$$\tilde{M}_0, \quad \|\tilde{M}_0\|^2 - \int_0^{\cdot} \|\sigma(\tilde{X}_0(s))\|^2\,\mathrm{d}s, \quad \tilde{M}_0 \otimes \tilde{W}_0 - \int_0^{\cdot} \sigma(\tilde{X}_0(s))\,\mathrm{d}s$$

are martingales, in other words,

$$\left\langle \tilde{M}_0 - \int_0^{\cdot} \sigma(\tilde{X}_0(s))\,\mathrm{d}\tilde{W}_0(s) \right\rangle = 0,$$

whence one concludes that $(\tilde{W}_0, \tilde{X}_0)$ is a weak solution. If the additional integrability hypothesis on φ is not satisfied, the proof remains almost the same, only a suitable cut-off procedure must be amended.

We take a step further and eliminate also the Skorokhod representation theorem. Let $\tilde{\mathbb{P}}_k$ be the laws of (X_{r_k}, W) on the space $U = \mathscr{C}([0,T];\mathbb{R}^m) \times \mathscr{C}([0,T];\mathbb{R}^n)$; we know that the sequence $\{\tilde{\mathbb{P}}_k\}$ converges weakly to some measure $\tilde{\mathbb{P}}_0$. Denote by (Y, B) the canonical process on U and set

$$\bar{M}_k = Y - Y(0) - \int_0^{\cdot} b_{r_k}(Y(s))\,\mathrm{d}s, \quad k \geq 0$$

(with $b_{r_0} = b$, $\sigma_{r_0} = \sigma$). Then

$$\bar{M}_k, \quad \|\bar{M}_k\|^2 - \int_0^{\cdot} \|\sigma_{r_k}(Y(s))\|^2\,\mathrm{d}s, \quad \bar{M}_k \otimes B - \int_0^{\cdot} \sigma_{r_k}(Y(s))\,\mathrm{d}s, \tag{5.5}$$

are local martingales under the measure $\tilde{\mathbb{P}}_k$ for every $k \geq 1$, as can be inferred quite easily from the definition of the measure $\tilde{\mathbb{P}}_k$. Now one may try to use Theorem IX.1.17 from [41] stating, roughly speaking, that a limit in law of a sequence of continuous local martingales is a local martingale. We do not use this theorem explicitly, since to establish convergence in law of the processes (5.5) as $k \to \infty$ is not simpler than to check the local martingale property for $k = 0$ directly, but our argument is inspired by the proofs in the book [41]. The proof we propose is not difficult and it is almost self-contained, it requires only two auxiliary lemmas (with simple proofs) from [41] on continuity properties of certain first entrance times which we recall in Appendix. Once we know that the processes (5.5) are local martingales for $k = 0$ as well, the trick from [11] and [60] may be used yielding that (B, Y) is a weak solution to (5.1), (5.2). It is worth mentioning that this procedure is independent of any integrability hypothesis on φ.

The proof of (5.3) not being our main concern notwithstanding, we decided to include a less standard proof of tightness inspired also by the theory of stochastic partial differential equations. We adopt an argument proposed by D. Gątarek and B. Gołdys in [27] (cf. also [15], Chapter 8), who introduced it when studying weak solutions to stochastic evolution equations in Hilbert spaces, and which relies on the factorization method of G. Da Prato, S. Kwapień and J. Zabczyk (see [15], Chapters 5 and 7, for a thorough exposition) and on compactness properties of fractional integral operators. The fractional calculus has become popular amongst probabilists recently because of its applications to fractional Brownian motion driven stochastic integrals and a proof of tightness using it may suit some readers more than the traditional one based on estimates of moduli of continuity.

Let us close this Introduction by stating the result to be proved precisely.

Theorem 5.1.1. *Let $b : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ and $\sigma : [0,T] \times \mathbb{R}^m \to \mathbb{M}_{m\times n}$ be Borel functions such that $b(t,\cdot)$ and $\sigma(t,\cdot)$ are continuous on $\mathbb{R}^m$ for any $t \in [0,T]$ and the linear growth hypothesis is satisfied, that is*

$$\exists K_* < \infty \ \forall t \in [0,T] \ \forall x \in \mathbb{R}^m \quad \|b(t,x)\| \vee \|\sigma(t,x)\| \le K_*\big(1 + \|x\|\big). \tag{5.6}$$

Let ν be a Borel probability measure on $\mathbb{R}^m$. Then there exists a weak solution to the problem

$$\mathrm{d}X = b(t,X)\,\mathrm{d}t + \sigma(t,X)\,\mathrm{d}W, \qquad X(0) \overset{d}{\sim} \nu. \tag{5.7}$$

We recall that a weak solution to (5.7) is a triple $((G, \mathscr{G}, (\mathscr{G}_t), \mathbb{Q}), W, X)$, where $(G, \mathscr{G}, (\mathscr{G}_t), \mathbb{Q})$ is a stochastic basis with a filtration $(\mathscr{G}_t)$ that satisfies the usual conditions, W is an n-dimensional $(\mathscr{G}_t)$-Wiener process and X is an $\mathbb{R}^m$-valued $(\mathscr{G}_t)$-progressively measurable process such that $\mathbb{Q} \circ X(0)^{-1} = \nu$ and

$$X(t) = X(0) + \int_0^t b(r, X(r))\,\mathrm{d}r + \int_0^t \sigma(r, X(r))\,\mathrm{d}W(r)$$

for all $t \in [0,T]$ $\mathbb{Q}$-almost surely.

The rest of the paper is devoted to the proof of Theorem 5.1.1. In Section 5.2, a sequence of equations with Lipschitzian coefficients approximation (5.7) is constructed, tightness of the set of their solutions being shown in Section 5.3. In Section 5.4, cluster points of the set of approximating solutions are identified as weak solutions to (5.7).

5.2 Approximations

In this section we introduce a sequence of equations which have strong solutions and approximate the problem (5.7). If E and F are metric spaces, we denote by $\mathscr{C}(E;F)$ the space of all continuous mappings from E to F. For brevity, we shall sometimes write $\mathscr{C}_V$ instead of $\mathscr{C}([0,T];\mathbb{R}^V)$ if $V \in \mathbb{N}$. If $f \in \mathscr{C}([0,T];F)$ and $s \in [0,T]$ then the restriction of f to the interval $[0,s]$ will be denoted by $\varrho_s f$. Plainly, $\varrho_s : \mathscr{C}([0,T];F) \to \mathscr{C}([0,s];F)$ is a continuous mapping. Finally, $L^q(G;\mathbb{R}^V)$ stands for the space of q-integrable functions on G with values in $\mathbb{R}^V$.

Our construction is based on the following proposition.

Proposition 5.2.1. *Suppose that $F : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^V$ is a Borel function of at most linear growth, i.e.*

$$\exists L < \infty \ \forall t \geq 0 \ \forall x \in \mathbb{R}^N \quad \|F(t,x)\| \leq L\big(1 + \|x\|\big),$$

such that $F(t,\cdot) \in \mathscr{C}(\mathbb{R}^N;\mathbb{R}^V)$ for any $t \in \mathbb{R}_+$. Then there exists a sequence of Borel functions $F_k : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^V$, $k \geq 1$, which have at most linear growth uniformly in k, namely

$$\forall k \geq 1 \ \forall t \geq 0 \ \forall x \in \mathbb{R}^N \quad \|F_k(t,x)\| \leq L\big(2 + \|x\|\big),$$

which are Lipschitz continuous in the second variable uniformly in the first one,

$$\forall k \geq 1 \ \exists L_k < \infty \ \forall t \geq 0 \ \forall x, y \in \mathbb{R}^N \quad \|F_k(t,x) - F_k(t,y)\| \leq L_k\|x - y\|,$$

and which satisfy

$$\lim_{k\to\infty} F_k(t,\cdot) = F(t,\cdot) \quad \textit{locally uniformly on } \mathbb{R}^N$$

for all $t \geq 0$.

Proof. The proof is rather standard so it is not necessary to dwell on its details: one takes a smooth function $\zeta \in \mathscr{C}^\infty(\mathbb{R}^N)$ such that $\zeta \geq 0$, $\operatorname{supp}\zeta \subseteq \{x \in \mathbb{R}^N;\ \|x\| \leq 1\}$ and $\int_{\mathbb{R}^N} \zeta \,\mathrm{d}x = 1$ and sets

$$G_k(t,x) = k^N \int_{\mathbb{R}^N} F(t,y)\zeta\big(k(x-y)\big)\,\mathrm{d}y$$

for $k \geq 1$, $t \geq 0$ and $x \in \mathbb{R}^N$. The functions G_k have all desired properties except for being only locally Lipschitz, but it is possible to modify them outside a sufficiently large ball in an obvious manner. □

Let the coefficients b and σ satisfy the assumptions of Theorem 5.1.1. Using Proposition 5.2.1 we find Borel functions $b_k : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ and $\sigma_k : [0,T] \times \mathbb{R}^m \to \mathbb{M}_{m\times n}$, $k \geq 1$, such that

$$\sup_{k\geq 1} \sup_{t\in[0,T]} \big(\|b_k(t,x)\| \vee \|\sigma_k(t,x)\|\big) \leq K_*\big(2 + \|x\|\big), \quad x \in \mathbb{R}^m, \tag{5.8}$$

$b_k(t,\cdot)$ and $\sigma_k(t,\cdot)$ are Lipschitz continuous uniformly in $t \in [0,T]$ and converge locally uniformly on $\mathbb{R}^m$ as $k \to \infty$ to $b(t,\cdot)$ and $\sigma(t,\cdot)$, respectively, for all $t \in [0,T]$. Fix an arbitrary stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$, on which an n-dimensional $(\mathscr{F}_t)$-Wiener process W and an $\mathscr{F}_0$-measurable random variable $\varphi : \Omega \to \mathbb{R}^m$ having the law ν are defined. It is well known that for any $k \geq 1$ there exists a unique $(\mathscr{F}_t)$-progressively measurable $\mathbb{R}^m$-valued stochastic process X_k solving the equation

$$\mathrm{d}X_k = b_k(t, X_k)\,\mathrm{d}t + \sigma_k(t, X_k)\,\mathrm{d}W, \quad X_k(0) = \varphi. \tag{5.9}$$

Moreover, for any $p \in [2, \infty)$ there exists a constant $C_* < \infty$, depending only on p, T and K_*, such that

$$\sup_{k \geq 1} \mathbb{E} \sup_{0 \leq t \leq T} \|X_k(t)\|^p \leq C_* \big(1 + \mathbb{E}\|\varphi\|^p\big), \tag{5.10}$$

provided that

$$\int_{\mathbb{R}^m} \|x\|^p \,\mathrm{d}\nu(x) = \mathbb{E}\|\varphi\|^p < \infty.$$

5.3 Tightness

Let $\{X_k;\ k \geq 1\}$ be the sequence of solutions to (5.9). Plainly, the processes X_k may be viewed as random variables $X_k : \Omega \to \mathscr{C}_m$ (where the Polish metric space $\mathscr{C}_m$ is endowed with its Borel σ-algebra). In this section, we aim at establishing the following proposition.

Proposition 5.3.1. *The set $\{\mathbb{P} \circ X_k^{-1};\ k \geq 1\}$ of Borel probability measures on*

$$\mathscr{C}([0, T]; \mathbb{R}^m)$$

is tight.

To this end, let us recall the definition of the Riemann-Liouville (or fractional integral) operator: if $q \in]1, \infty]$, $\alpha \in]\frac{1}{q}, 1]$ and $f \in L^q([0, T]; \mathbb{R}^m)$, we define a function $R_\alpha f : [0, T] \to \mathbb{R}^m$ by

$$\big(R_\alpha f\big)(t) = \int_0^t (t - s)^{\alpha - 1} f(s)\,\mathrm{d}s, \quad 0 \leq t \leq T.$$

The definition is correct, as an easy application of the Hölder inequality shows. Note that, in particular, $R_1 f = \int_0^{\cdot} f(t)\,\mathrm{d}t$. It is well-known (and may be checked by very straightforward calculations) that R_α is a bounded linear operator from $L^q([0, T]; \mathbb{R}^m)$ to $\mathscr{C}^{0, \alpha - 1/q}([0, T]; \mathbb{R}^m)$, the space of $(\alpha - \frac{1}{q})$-Hölder continuous functions (see e.g. [69], Theorem 3.6). Balls in $\mathscr{C}^{0, \alpha - 1/q}([0, T]; \mathbb{R}^m)$ are relatively compact in $\mathscr{C}([0, T]; \mathbb{R}^m)$ by the Arzelà-Ascoli theorem, hence we arrive at

Lemma 5.3.2. *If $q \in]1, \infty]$ and $\alpha \in]\frac{1}{q}, 1]$, then R_α is a compact linear operator from $L^q([0, T]; \mathbb{R}^m)$ to $\mathscr{C}([0, T]; \mathbb{R}^m)$.*

We shall need also a Fubini-type theorem for stochastic integrals in the following form (a more general result may be found in [15], Theorem 4.18):

Lemma 5.3.3. *Let (X, Σ, μ) be a finite measure space, $(G, \mathscr{G}, (\mathscr{G}_t), \mathbb{Q})$ a stochastic basis, and B an n-dimensional $(\mathscr{G}_t)$-Wiener process. Denote by $\mathscr{M}$ the σ-algebra of $(\mathscr{G}_t)$-progressively measurable sets and assume that $\psi : [0, T] \times G \times X \to \mathbb{M}_{m \times n}$ is an $\mathscr{M} \otimes \Sigma$-measurable mapping such that*

$$\int_X \left(\int_0^T \int_G \|\psi(s, x)\|^2 \,\mathrm{d}\mathbb{Q}\,\mathrm{d}s \right)^{1/2} \mathrm{d}\mu(x) < \infty. \tag{5.11}$$

Then

$$\int_X \left[\int_0^T \psi(s,x)\,\mathrm{d}B(s)\right]\mathrm{d}\mu(x) = \int_0^T \left[\int_X \psi(s,x)\,\mathrm{d}\mu(x)\right]\mathrm{d}B(s)$$

$\mathbb{Q}$*-almost surely.*

The last auxiliary result to be recalled is the Young inequality for convolutions (see, for example, [52], Theorem 4.2).

Lemma 5.3.4. *Let $p,r,s \in [1,\infty]$ satisfy*

$$\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{s}.$$

If $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, then the integral

$$(f*g)(x) \equiv \int_{\mathbb{R}^d} f(x-y)g(y)\,\mathrm{d}y$$

converges for almost all $x \in \mathbb{R}^d$, $f*g \in L^s(\mathbb{R}^d)$ and

$$\big\|f*g\big\|_{L^s} \le \|f\|_{L^p}\|g\|_{L^q}.$$

In fact, we shall need only a particular one-dimensional case of Lemma 5.3.4: if $f \in L^p(0,T)$, $g \in L^q(0,T)$, $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{s}$, then

$$\int_0^T \left|\int_0^t f(t-r)g(r)\,\mathrm{d}r\right|^s \mathrm{d}t \le \|f\|^s_{L^p(0,T)}\|g\|^s_{L^q(0,T)}. \tag{5.12}$$

Now we derive a representation formula that plays a key role in our proof of Proposition 5.3.1.

Lemma 5.3.5. *Let ψ be an $\mathbb{M}_{m\times n}$-valued progressively measurable process such that*

$$\mathbb{E}\int_0^T \|\psi(s)\|^q\,\mathrm{d}s < \infty$$

for some $q>2$. Choose $\alpha \in \left]\frac{1}{q},\frac{1}{2}\right[$ and set

$$Z(t) = \int_0^t (t-u)^{-\alpha}\psi(u)\,\mathrm{d}W(u), \quad 0 \le t \le T.$$

Then

$$\int_0^t \psi(s)\,\mathrm{d}W(s) = \frac{\sin\pi\alpha}{\pi}\big(R_\alpha Z\big)(t)$$

for all $t \in [0,T]$ $\mathbb{P}$-almost surely.

Proof. The result is well-known and widely used for infinite-dimensional systems (see e.g. [15], § 5.3). For finite-dimensional equations, the proof is slightly simpler and thus it is repeated here for the reader's convenience.

Since $s^{-2\alpha} \in L^1(0,T)$, $\mathbb{E}\|\psi(\cdot)\|^2 \in L^1(0,T)$, their convolution

$$t \longmapsto \int_0^t (t-s)^{-2\alpha}\mathbb{E}\|\psi(s)\|^2\,\mathrm{d}s = \mathbb{E}\int_0^t \big|(t-s)^{-\alpha}\|\psi(s)\|\big|^2\,\mathrm{d}s$$

belongs to $L^1(0,T)$ as well and so is finite almost everywhere in $[0,T]$, which implies that $Z(t)$ is well defined for almost all $t \in [0,T]$. By the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}
\mathbb{E}\int_0^T \|Z(t)\|^q \,\mathrm{d}t &= \int_0^T \mathbb{E}\left\|\int_0^s (s-u)^{-\alpha}\psi(u)\,\mathrm{d}W(u)\right\|^q \mathrm{d}s \\
&\le C_q \mathbb{E}\int_0^T \left(\int_0^s (s-u)^{-2\alpha}\|\psi(u)\|^2 \,\mathrm{d}u\right)^{q/2} \mathrm{d}s \\
&\le C_q \left(\int_0^T s^{-2\alpha}\,\mathrm{d}s\right)^{q/2} \left(\int_0^T \mathbb{E}\|\psi(u)\|^q \,\mathrm{d}u\right),
\end{aligned}$$

the last estimate being a consequence of (5.12) and the fact that $\mathbb{E}\|\psi(\cdot)\|^2 \in L^{q/2}(0,T)$. Hence $Z(\cdot,\omega) \in L^q(0,T;\mathbb{R}^m)$ for $\mathbb{P}$-almost all $\omega \in \Omega$ and $R_\alpha Z$ is well defined $\mathbb{P}$-almost surely. Further,

$$\begin{aligned}
&\int_0^t \left(\mathbb{E}\int_0^t \left\|(t-s)^{\alpha-1}\mathbf{1}_{[0,s[}(u)(s-u)^{-\alpha}\psi(u)\right\|^2 \mathrm{d}u\right)^{1/2} \mathrm{d}s \\
&\quad= \int_0^t (t-s)^{\alpha-1}\left(\int_0^s (s-u)^{-2\alpha}\mathbb{E}\|\psi(u)\|^2 \,\mathrm{d}u\right)^{1/2} \mathrm{d}s \\
&\quad\le \left(\int_0^t s^{(\alpha-1)q^*}\,\mathrm{d}s\right)^{1/q^*} \left(\int_0^t \left(\int_0^s (s-u)^{-2\alpha}\mathbb{E}\|\psi(u)\|^2 \,\mathrm{d}u\right)^{q/2} \mathrm{d}s\right)^{1/q} \\
&\quad\le \left(\int_0^t s^{(\alpha-1)q^*}\,\mathrm{d}s\right)^{1/q^*} \left(\int_0^t s^{-2\alpha}\,\mathrm{d}s\right)^{1/2} \left(\int_0^t \mathbb{E}\|\psi(u)\|^q \,\mathrm{d}u\right)^{1/q} < \infty,
\end{aligned}$$

where $\frac{1}{q^*}+\frac{1}{q}=1$ and the Hölder and Young inequalities were used consecutively. This means that the hypothesis (5.11) of Lemma 5.3.3 is satisfied and this lemma may be used to obtain

$$\begin{aligned}
\big(R_\alpha Z\big)(t) &= \int_0^t (t-s)^{\alpha-1}\left(\int_0^s (s-u)^{-\alpha}\psi(u)\,\mathrm{d}W(u)\right)\mathrm{d}s \\
&= \int_0^t\int_0^t (t-s)^{\alpha-1}\mathbf{1}_{[0,s[}(u)(s-u)^{-\alpha}\psi(u)\,\mathrm{d}W(u)\,\mathrm{d}s \\
&= \int_0^t \left[\int_0^t (t-s)^{\alpha-1}\mathbf{1}_{[0,s[}(u)(s-u)^{-\alpha}\,\mathrm{d}s\right]\psi(u)\,\mathrm{d}W(u) \\
&= \int_0^t \left[\int_u^t (t-s)^{\alpha-1}(s-u)^{-\alpha}\,\mathrm{d}s\right]\psi(u)\,\mathrm{d}W(u) \\
&= \int_0^t \underbrace{\left[\int_0^1 (1-v)^{\alpha-1}v^{-\alpha}\,\mathrm{d}v\right]}_{=\frac{\pi}{\sin\pi\alpha}}\psi(u)\,\mathrm{d}W(u).
\end{aligned}$$

□

Proof of Proposition 5.3.1. Let an arbitrary $\varepsilon > 0$ be given, we have to find a relatively compact set $K \subseteq \mathscr{C}_m$ such that

$$\inf_{k\ge 1}\mathbb{P}\big\{X_k \in K\big\} \ge 1-\varepsilon.$$

In what follows, we shall denote by D_i constants independent of k and by $|\cdot|_q$ the norm of $L^q(0,T;\mathbb{R}^m)$.

First, we prove our claim under an additional assumption that there exists $p > 2$ such that

$$\mathbb{E}\|\varphi\|^p < \infty. \tag{5.13}$$

Plainly, a compact set $\Gamma \subseteq \mathbb{R}^m$ may be found satisfying

$$\nu(\Gamma) = \mathbb{E}\{\varphi \in \Gamma\} \geq 1 - \frac{\varepsilon}{3}.$$

Take an $\alpha \in]\frac{1}{p}, \frac{1}{2}[$. By Lemma 5.3.5,

$$\begin{aligned} X_k(t) &= \varphi + \int_0^t b_k(s, X_k(s))\,\mathrm{d}s + \int_0^t \sigma_k(s, X_k(s))\,\mathrm{d}W(s) \\ &= \varphi + \big[R_1 b(\cdot, X_k(\cdot))\big](t) + \frac{\sin \pi\alpha}{\pi}\big(R_\alpha Z_k\big)(t), \quad 0 \leq t \leq T, \end{aligned}$$

$\mathbb{P}$-almost surely, where

$$Z_k(s) = \int_0^s (s-u)^{-\alpha}\sigma_k(u, X_k(u))\,\mathrm{d}W(u), \quad 0 \leq s \leq T.$$

Applying the Chebyshev inequality, (5.8) and (5.10) we get

$$\begin{aligned} \mathbb{P}\big\{|b_k(\cdot, X_k(\cdot))|_p \geq \Lambda\big\} &\leq \frac{1}{\Lambda^p}\mathbb{E}\int_0^T \big\|b_k(t, X_k(t))\big\|^p\,\mathrm{d}t \\ &\leq \frac{1}{\Lambda^p}K_*^p\mathbb{E}\int_0^T \big(2 + \|X_k(t)\|\big)^p\,\mathrm{d}t \\ &\leq \frac{D_1}{\Lambda^p}\big(1 + \mathbb{E}\|\varphi\|^p\big). \end{aligned}$$

Similarly, invoking in addition the Burkholder-Davis-Gundy and Young inequalities,

$$\begin{aligned} \mathbb{P}\big\{|Z_k|_p \geq \Lambda\big\} &\leq \frac{1}{\Lambda^p}\mathbb{E}\int_0^T \|Z_k(t)\|^p\,\mathrm{d}t \\ &\leq \frac{D_2}{\Lambda^p}\mathbb{E}\int_0^T \bigg(\int_0^t (t-s)^{-2\alpha}\big\|\sigma_k(s, X_k(s))\big\|^2\,\mathrm{d}s\bigg)^{p/2}\,\mathrm{d}t \\ &\leq \frac{D_2}{\Lambda^p}\bigg(\int_0^T s^{-2\alpha}\,\mathrm{d}s\bigg)^{p/2}\bigg(\int_0^T \mathbb{E}\big\|\sigma_k(s, X_k(s))\big\|^p\,\mathrm{d}s\bigg) \\ &\leq \frac{D_3}{\Lambda^p}\big(1 + \mathbb{E}\|\varphi\|^p\big). \end{aligned}$$

Let us choose $\Lambda_0 < \infty$ so that

$$\frac{D_1 + D_3}{\Lambda_0^p}\big(1 + \mathbb{E}\|\varphi\|^p\big) < \frac{\varepsilon}{3}$$

and set

$$\begin{aligned} K = \Big\{ f \in \mathscr{C}([0,T]; \mathbb{R}^m);\ f = x + R_1 r + \frac{\sin \pi\alpha}{\pi}R_\alpha v,\ x \in \Gamma, \\ r, v \in L^p(0,T; \mathbb{R}^m),\ |r|_p \vee |v|_p \leq \Lambda_0 \Big\}. \end{aligned}$$

Since the operators R_1 and R_α are compact, the set K is relatively compact and

$$\begin{aligned}\mathbb{P}\{X_k \notin K\} &\leq \mathbb{P}\{\varphi \notin \Gamma\} + \mathbb{P}\{|b_k(\cdot, X_k(\cdot))|_p > \Lambda_0\} + \mathbb{P}\{|Z_k|_p > \Lambda_0\}\\ &\leq \frac{2}{3}\varepsilon < \varepsilon\end{aligned}$$

for any $k \geq 1$, which completes the proof of tightness under the additional assumption (5.13).

Finally, let φ be arbitrary. Let $\varepsilon > 0$ be fixed, we may find $\Pi \geq 0$ such that $\mathbb{P}\{\|\varphi\| > \Pi\} < \frac{\varepsilon}{2}$. Let $\hat{X}_k$, $k \geq 1$, be the solutions to

$$\mathrm{d}\hat{X}_k = b_k(t, \hat{X}_k)\,\mathrm{d}t + \sigma_k(t, \hat{X}_k)\,\mathrm{d}W, \quad \hat{X}_k(0) = \mathbf{1}_{\{\|\varphi\| \leq \Pi\}}\varphi. \tag{5.14}$$

The initial condition in (5.14) satisfies (5.13), so by the first part of the proof we know that the set $\{\mathbb{P} \circ \hat{X}_k^{-1};\ k \geq 1\}$ is tight and there exists a compact set $K \subseteq \mathscr{C}_m$ such that

$$\inf_{k \geq 1} \mathbb{P}\{\hat{X}_k \notin K\} \leq \frac{\varepsilon}{2}.$$

Since the coefficients b_k, σ_k are Lipschitz continuous in space variables,

$$\mathbf{1}_{\{\|\varphi\| \leq \Pi\}}\hat{X}_k = \mathbf{1}_{\{\|\varphi\| \leq \Pi\}}X_k \quad \mathbb{P}\text{-almost surely}$$

for all $k \geq 1$, this implies

$$\mathbb{P}\{X_k \notin K\} \leq \mathbb{P}\{\hat{X}_k \notin K\} + \mathbb{P}\{\|\varphi\| > \Pi\} < \varepsilon$$

for any $k \geq 1$ and tightness of the set $\{\mathbb{P} \circ X_k^{-1};\ k \geq 1\}$ follows. □

Corollary 5.3.6. *The set $\{\mathbb{P} \circ (X_k, W)^{-1};\ k \geq 1\}$ is a tight set of probability measures on $\mathscr{C}([0,T];\mathbb{R}^m) \times \mathscr{C}([0,T];\mathbb{R}^n)$.*

By the Prokhorov theorem, the set $\{\mathbb{P} \circ (X_k, W)^{-1};\ k \geq 1\}$ is relatively (sequentially) compact in the weak topology of probability measures, so it contains a weakly convergent subsequence. Without loss of generality we may (and shall) assume that the sequence $\{\mathbb{P} \circ (X_k, W)^{-1}\}_{k=1}^\infty$ itself is weakly convergent. Let us set for brevity $\tilde{\mathbb{P}}_k = \mathbb{P} \circ (X_k, W)^{-1}$, $k \geq 1$, and denote the weak limit of $\{\tilde{\mathbb{P}}_k\}_{k=1}^\infty$ by $\tilde{\mathbb{P}}_0$. Set further

$$U = \mathscr{C}_m \times \mathscr{C}_n, \quad \mathscr{U} = \mathrm{Borel}(\mathscr{C}_m) \otimes \mathrm{Borel}(\mathscr{C}_n),$$

and let (Y, B) be the process of projections on U, that is

$$(Y_t, B_t) : \mathscr{C}_m \times \mathscr{C}_n \longrightarrow \mathbb{R}^m \times \mathbb{R}^n,\ (h, g) \longmapsto (h(t), g(t)), \quad 0 \leq t \leq T.$$

Finally, let $(\mathscr{U}_t)$ be the $\tilde{\mathbb{P}}_0$-augmented canonical filtration of the process (Y, B), that is

$$\mathscr{U}_t = \sigma\big(\sigma(\varrho_t Y, \varrho_t B) \cup \{N \in \mathscr{U};\ \tilde{\mathbb{P}}_0(N) = 0\}\big),\ 0 \leq t \leq T.$$

5.4 Identification of the limit

In this section we shall show that $((U, \mathscr{U}, (\mathscr{U}_t), \tilde{\mathbb{P}}_0), B, Y)$ is a weak solution to the problem (5.7). Towards this end, define

$$M_k = Y - Y(0) - \int_0^{\cdot} b_k(r, Y(r))\,\mathrm{d}r, \quad k \geq 0,$$

where we set $b_0 = b$, $\sigma_0 = \sigma$. The proof is an immediate consequence of the following four lemmas.

Lemma 5.4.1. *The process M_0 is an m-dimensional local $(\mathscr{U}_t)$-martingale on $(U, \mathscr{U}, \tilde{\mathbb{P}}_0)$.*

Lemma 5.4.2. *The process B is an n-dimensional $(\mathscr{U}_t)$-Wiener process on $(U, \mathscr{U}, \tilde{\mathbb{P}}_0)$.*

Lemma 5.4.3. *The process*

$$\|M_0\|^2 - \int_0^{\cdot} \big\|\sigma(r, Y(r))\big\|^2 \, \mathrm{d}r$$

is a local $(\mathscr{U}_t)$-martingale on $(U, \mathscr{U}, \tilde{\mathbb{P}}_0)$.

Lemma 5.4.4. *The process*

$$M_0 \otimes B - \int_0^{\cdot} \sigma(r, Y(r)) \, \mathrm{d}r$$

is an $\mathbb{M}_{m\times n}$-valued local $(\mathscr{U}_t)$-martingale on $(U, \mathscr{U}, \tilde{\mathbb{P}}_0)$.

Proofs of these lemmas have an identical structure, so we prove only the first of them in detail, the other ones being treated only in a concise manner. In the course of the proof, we shall need two easy results on continuity properties of the first entrance times as functionals of paths. Let $V \geq 1$, for any $L \in \mathbb{R}_+$ define

$$\tau_L : \mathscr{C}_V \longrightarrow [0, T], \;\; f \longmapsto \inf\big\{t \geq 0;\; \|f(t)\| \geq L\big\}$$

(with a convention $\inf \emptyset = T$).

Lemma 5.4.5. *The following holds true*

(i) for any $f \in \mathscr{C}_V$, the function $L \mapsto \tau_L(f)$ is nondecreasing and left-continuous on $\mathbb{R}_+$,

(ii) for each $L \in \mathbb{R}_+$, the mapping τ_L is lower semicontinuous. Moreover, τ_L is continuous at every point $f \in \mathscr{C}_V$ for which $\tau_\bullet(f)$ is continuous at L.

If $(Z_t)_{t\in[0,T]}$ is a continuous $\mathbb{R}^V$-valued stochastic process defined on a probability space $(G, \mathscr{G}, \boldsymbol{q})$, then $\big(\tau_L(Z)\big)_{L\geq 0}$ is a stochastic process with nondecreasing left-continuous trajectories, whence we get

Lemma 5.4.6. *The set*

$$\big\{L \in \mathbb{R}_+;\; \boldsymbol{q}\{\tau_\bullet(Z) \text{ is not continuous at } L\} > 0\big\}$$

is at most countable.

Lemma 5.4.5 is proved (but not stated exactly in this form) in [41], see Lemma VI.2.10 and Proposition VI.2.11 there. For Lemma 5.4.6, see [41], Lemma VI.3.12. In the book [41], τ_L is considered as a function on the Skorokhod space $\mathbb{D}$, in our case the proofs simplify further; they are recalled in Appendix to keep the paper self-contained.

Further, let us quote an useful result on weak convergence of measures (cf. e.g. [8], Proposition IX.5.7).

Lemma 5.4.7. *Let $\{\nu_r\}_{r\geq 1}$ be a sequence of Borel probability measures on a metric space Θ converging weakly to a Borel probability measure ν_0. Let $f:\Theta\to\mathbb{R}$ be a bounded real function continuous at ν_0-almost all points of Θ. Then*

$$\lim_{r\to\infty}\int_\Theta f\,\mathrm{d}\nu_r = \int_\Theta f\,\mathrm{d}\nu_0.$$

Proof of Lemma 5.4.1. The idea of the proof is simple: define processes

$$\mu_k = X_k - X_k(0) - \int_0^\cdot b_k(r, X_k(r))\,\mathrm{d}r, \quad k\geq 1,$$

in analogy with the definition of M_k but using the solutions X_k to the problem (5.9) instead of the process Y. We shall prove: i) μ_k, $k\geq 1$, are local martingales, ii) M_k, $k\geq 1$, are local martingales with respect to the measure $\tilde{\mathbb{P}}_k$ due to the equality of laws $\tilde{\mathbb{P}}_k\circ(Y,B)^{-1} = \mathbb{P}\circ(X_k,W)^{-1}$, iii) M_0 is a local martingale as a limit of local martingales M_k.

First, as X_k solves (5.9),

$$\mu_k(t) = \int_0^t \sigma_k(r, X_k(r))\,\mathrm{d}W_r, \quad 0\leq t\leq T,$$

and so μ_k is a local $(\mathscr{F}_t)$-martingale. Take an $L\in\mathbb{R}_+$, for the time being arbitrary. Obviously, $\tau_L(X_k)$ is a stopping time and $\mu_k(\cdot\wedge\tau_L(X_k))$ is a bounded process by (5.8) and the definition of τ_L, hence $\mu_k(\cdot\wedge\tau_L(X_k))$ is a martingale.

Hereafter, times $s,t\in[0,T]$, $s\leq t$, and a continuous function

$$\gamma:\mathscr{C}([0,s];\mathbb{R}^m)\times\mathscr{C}([0,s];\mathbb{R}^n)\longrightarrow[0,1]$$

will be fixed but otherwise arbitrary. Obviously, $\gamma(\varrho_s X_k, \varrho_s W)$ is a bounded $\mathscr{F}_s$-measurable function, hence

$$\mathbb{E}\gamma(\varrho_s X_k,\varrho_s W)\mu_k(t\wedge\tau_L(X_k)) = \mathbb{E}\gamma(\varrho_s X_k,\varrho_s W)\mu_k(s\wedge\tau_L(X_k)) \tag{5.15}$$

by the martingale property of $\mu_k(\cdot\wedge\tau_L(X_k))$.

Note that the mapping

$$[0,T]\times\mathscr{C}_m\longrightarrow\mathbb{R}^m,\ (u,h)\longmapsto h(u)-h(0)-\int_0^u b_k(r,h(r))\,\mathrm{d}r$$

is continuous for any $k\geq 0$ due to the continuity of $b_k(r,\cdot)$, and the mapping

$$\mathscr{C}_m\longrightarrow[0,T]\times\mathscr{C}_m,\ h\longmapsto(\xi\wedge\tau_L(h),h)$$

is Borel for any $\xi\in[0,T]$ fixed by Lemma 5.4.5(ii), thus also their superposition

$$H_k(\xi,\cdot):\mathscr{C}_m\longrightarrow\mathbb{R}^m,\ h\longmapsto h(\xi\wedge\tau_L(h))-h(0)-\int_0^{\xi\wedge\tau_L(h)} b_k(r,h(r))\,\mathrm{d}r$$

is Borel. Consequently, the mapping

$$\mathscr{C}_m\times\mathscr{C}_n\longrightarrow\mathbb{R}^m,\ (h,g)\longmapsto\gamma(\varrho_s h,\varrho_s g)H_k(\xi,h)$$

is Borel. Since $\mu_k(\xi \wedge \tau_L(X_k)) = H_k(\xi, X_k)$, $k \geq 1$, and $M_k(\xi \wedge \tau_L(Y)) = H_k(\xi, Y)$, $k \geq 0$, we get

$$\mathbb{P} \circ \big[\gamma(\varrho_s X_k, \varrho_s W)\mu_k(\xi \wedge \tau_L(X_k))\big]^{-1} = \tilde{\mathbb{P}}_k \circ \big[\gamma(\varrho_s Y, \varrho_s B) M_k(\xi \wedge \tau_L(Y))\big]^{-1}$$

for all $k \geq 1$ by the definition of $\tilde{\mathbb{P}}_k$, which together with (5.15) implies

$$\tilde{\mathbb{E}}_k \gamma(\varrho_s Y, \varrho_s B) M_k(t \wedge \tau_L(Y)) = \tilde{\mathbb{E}}_k \gamma(\varrho_s Y, \varrho_s B) M_k(s \wedge \tau_L(Y)), \ k \geq 1. \tag{5.16}$$

Now, suppose in addition that L is chosen so that

$$\tilde{\mathbb{P}}_0 \big\{\tau_\bullet(Y) \text{ is continuous at } L\big\} = 1. \tag{5.17}$$

(Lemma 5.4.6 shows that such a choice is possible.) Then

$$\tilde{\mathbb{P}}_0 \big\{(f,g) \in U;\ \tau_L(\cdot) \text{ is continuous at } f\big\} = 1$$

by Lemma 5.4.5(ii) and the fact that Y is a canonical projection from U onto $\mathscr{C}_m$, so also

$$\tilde{\mathbb{P}}_0 \big\{(f,g) \in U;\ H_0(\xi, \cdot) \text{ is continuous at } f\big\} = 1.$$

This implies that $\gamma(\varrho_s Y, \varrho_s B) H_0(\xi, Y)$ is a bounded function continuous $\tilde{\mathbb{P}}_0$-almost everywhere on U for any ξ fixed. We may estimate

$$\begin{aligned}
&\big\|\tilde{\mathbb{E}}_k \gamma(\varrho_s Y, \varrho_s B) H_k(\xi, Y) - \tilde{\mathbb{E}}_0 \gamma(\varrho_s Y, \varrho_s B) H_0(\xi, Y)\big\| \\
&\quad \leq \big\|\tilde{\mathbb{E}}_k \gamma(\varrho_s Y, \varrho_s B)\big[H_k(\xi, Y) - H_0(\xi, Y)\big]\big\| \\
&\qquad + \big\|\tilde{\mathbb{E}}_k \gamma(\varrho_s Y, \varrho_s B) H_0(\xi, Y) - \tilde{\mathbb{E}}_0 \gamma(\varrho_s Y, \varrho_s B) H_0(\xi, Y)\big\|.
\end{aligned}$$

From Lemma 5.4.7 we obtain that

$$\lim_{k \to \infty} \tilde{\mathbb{E}}_k \gamma(\varrho_s Y, \varrho_s B) H_0(\xi, Y) = \tilde{\mathbb{E}}_0 \gamma(\varrho_s Y, \varrho_s B) H_0(\xi, Y).$$

Further,

$$\begin{aligned}
&\Big\|\tilde{\mathbb{E}}_k \gamma(\varrho_s Y, \varrho_s B)\big[H_k(\xi, Y) - H_0(\xi, Y)\big]\Big\| \\
&\quad \leq \tilde{\mathbb{E}}_k \big\|H_k(\xi, Y) - H_0(\xi, Y)\big\| \\
&\quad = \tilde{\mathbb{E}}_k \bigg\|\int_0^{\xi \wedge \tau_L(Y)} \big[b_k(r, Y(r)) - b_0(r, Y(r))\big]\,\mathrm{d}r\bigg\| \\
&\quad = \tilde{\mathbb{E}}_k \mathbf{1}_{\{\tau_L(Y) > 0\}} \bigg\|\int_0^{\xi \wedge \tau_L(Y)} \big[b_k(r, Y(r)) - b_0(r, Y(r))\big]\,\mathrm{d}r\bigg\| \\
&\quad \leq \tilde{\mathbb{E}}_k \mathbf{1}_{\{\tau_L(Y) > 0\}} \int_0^{\xi \wedge \tau_L(Y)} \big\|b_k(r, Y(r)) - b_0(r, Y(r))\big\|\,\mathrm{d}r \\
&\quad \leq \tilde{\mathbb{E}}_k \mathbf{1}_{\{\tau_L(Y) > 0\}} \int_0^T \big\|b_k(r, Y(r \wedge \tau_L(Y))) - b_0(r, Y(r \wedge \tau_L(Y)))\big\|\,\mathrm{d}r \\
&\quad \leq \tilde{\mathbb{E}}_k \mathbf{1}_{\{\tau_L(Y) > 0\}} \int_0^T \sup_{\|z\| \leq L} \big\|b_k(r, z) - b_0(r, z)\big\|\,\mathrm{d}r \\
&\quad \leq \int_0^T \sup_{\|z\| \leq L} \big\|b_k(r, z) - b_0(r, z)\big\|\,\mathrm{d}r,
\end{aligned}$$

as $\|Y(r\wedge\tau_L(Y))\|\le L$ on the set $\{\tau_L(Y)>0\}$. Since $b_k(r,\cdot)\to b_0(r,\cdot)$ locally uniformly on $\mathbb{R}^m$ for every $r\in[0,T]$ and

$$\sup_{\|z\|\le L}\big\|b_k(r,z)-b_0(r,z)\big\|\le 2K_*(2+L)$$

by (5.6) and (5.8), we have

$$\lim_{k\to\infty}\int_0^T\sup_{\|z\|\le L}\big\|b_k(r,z)-b_0(r,z)\big\|\,\mathrm{d}r=0$$

by the dominated convergence theorem, hence

$$\lim_{k\to\infty}\tilde{\mathbb{E}}_k\gamma(\varrho_sY,\varrho_sB)H_k(\xi,Y)=\tilde{\mathbb{E}}_0\gamma(\varrho_sY,\varrho_sB)H_0(\xi,Y)$$

for any $\xi\in[0,T]$. Therefore,

$$\tilde{\mathbb{E}}_0\gamma(\varrho_sY,\varrho_sB)M_0(t\wedge\tau_L(Y))=\tilde{\mathbb{E}}_0\gamma(\varrho_sY,\varrho_sB)M_0(s\wedge\tau_L(Y))\tag{5.18}$$

follows from (5.16). If $G\subseteq\mathscr{C}([0,s];\mathbb{R}^m\times\mathbb{R}^n)$ is an arbitrary open set, then there exist continuous functions $g_l:\mathscr{C}([0,s];\mathbb{R}^m\times\mathbb{R}^n)\longrightarrow[0,1]$ such that $g_l\nearrow\mathbf{1}_G$ on $\mathscr{C}([0,s];\mathbb{R}^m\times\mathbb{R}^n)$ as $l\to\infty$. Therefore, using the Levi monotone convergence theorem we derive from (5.18) that

$$\tilde{\mathbb{E}}_0\mathbf{1}_G(\varrho_sY,\varrho_sB)M_0(t\wedge\tau_L(Y))=\tilde{\mathbb{E}}_0\mathbf{1}_G(\varrho_sY,\varrho_sB)M_0(s\wedge\tau_L(Y)).\tag{5.19}$$

Further,

$$\big\{G\subseteq\mathscr{C}([0,s];\mathbb{R}^m\times\mathbb{R}^n);\quad G\text{ Borel and (5.19) holds for }\mathbf{1}_G\big\}$$

is a λ-system containing, as we have just shown, the system of all open sets in

$$\mathscr{C}([0,s];\mathbb{R}^m\times\mathbb{R}^n)$$

closed under finite intersections. Consequently, (5.19) holds for all Borel sets $G\subseteq\mathscr{C}([0,s];\mathbb{R}^m\times\mathbb{R}^n)$, that is

$$\tilde{\mathbb{E}}_0\mathbf{1}_AM_0(t\wedge\tau_L(Y))=\tilde{\mathbb{E}}_0\mathbf{1}_AM_0(s\wedge\tau_L(Y))$$

holds for all $A\in\sigma(\varrho_sY,\varrho_sB)$, thus for all $A\in\mathscr{U}_s$. We see that $M_0(\cdot\wedge\tau_L(Y))$ is a $(\mathscr{U}_t)$-martingale, whenever $L\in\mathbb{R}_+$ satisfies (5.17). It remains to note that by Lemma 5.4.6 there exists a sequence $L_r\nearrow\infty$ such that

$$\tilde{\mathbb{P}}_0\big\{\tau_\bullet(Y)\text{ is continuous at }L_r\text{ for every }r\ge1\big\}=1.$$

As $\{\tau_{L_r}(Y)\}$ is plainly a localizing sequence of stopping times, we conclude that M_0 is a local $(\mathscr{U}_t)$-martingale on $(U,\mathscr{U},\tilde{\mathbb{P}}_0)$, as claimed. □

Proof of Lemma 5.4.2. By our construction, $\mathbb{P}\circ W^{-1}=\tilde{\mathbb{P}}_k\circ B^{-1}$ for each $k\ge1$, so also $\mathbb{P}\circ W^{-1}=\tilde{\mathbb{P}}_0\circ B^{-1}$ and B is an n-dimensional Wiener process (with respect to its canonical filtration) on $(U,\mathscr{U},\tilde{\mathbb{P}}_0)$. In particular, its tensor quadratic variation satisfies $\langle\!\langle B\rangle\!\rangle_t=tI$. Mimicking the procedure from the previous proof we may check easily that B is a local $(\mathscr{U}_t)$-martingale, hence an $(\mathscr{U}_t)$-Wiener process by the Lévy theorem. □

Proof of Lemma 5.4.3. We know that μ_k, $k \geq 1$, are local martingales and

$$\langle \mu_k \rangle = \left\langle \int_0^{\cdot} \sigma_k(r, X_k(r))\, \mathrm{d}W_r \right\rangle = \int_0^{\cdot} \left\|\sigma_k(r, X_k(r))\right\|^2 \mathrm{d}r,$$

thus

$$\|\mu_k\|^2 - \int_0^{\cdot} \left\|\sigma_k(r, X_k(r))\right\|^2 \mathrm{d}r, \quad k \geq 1,$$

are continuous local martingales. For times $s \leq t$ and a function γ introduced in the proof of Lemma 5.4.1 we get

$$\begin{aligned} &\mathbb{E}\gamma(\varrho_s X_k, \varrho_s W)\Big[\left\|\mu_k(t \wedge \tau_L(X_k))\right\|^2 - \int_0^{t \wedge \tau_L(X_k)} \left\|\sigma_k(r, X_k(r))\right\|^2 \mathrm{d}r\Big] \\ &\quad = \mathbb{E}\gamma(\varrho_s X_k, \varrho_s W)\Big[\left\|\mu_k(s \wedge \tau_L(X_k))\right\|^2 - \int_0^{s \wedge \tau_L(X_k)} \left\|\sigma_k(r, X_k(r))\right\|^2 \mathrm{d}r\Big]. \end{aligned} \tag{5.20}$$

Note that

$$\mathscr{C}_m \longrightarrow \mathbb{R},\ h \longmapsto \left\|H_k(\xi, h)\right\|^2 - \int_0^{\xi \wedge \tau_L(h)} \left\|\sigma_k(r, h(r))\right\|^2 \mathrm{d}r$$

is a Borel mapping for all $k \geq 0$ and $\xi \in [0, T]$. It can be seen easily that it suffices to check that

$$\mathscr{C}_m \longrightarrow \mathbb{R},\ h \longmapsto \int_0^u \left\|\sigma_k(r, h(r))\right\|^2 \mathrm{d}r$$

is a continuous mapping for any $u \in [0, T]$; this follows from the estimate

$$\begin{aligned} &\left| \int_0^u \left\|\sigma_k(r, h_1(r))\right\|^2 \mathrm{d}r - \int_0^u \left\|\sigma_k(r, h_2(r))\right\|^2 \mathrm{d}r \right| \\ &\leq \int_0^u \Big\{\left\|\sigma_k(r, h_1(r))\right\| + \left\|\sigma_k(r, h_2(r))\right\|\Big\} \Big|\left\|\sigma_k(r, h_1(r))\right\| - \left\|\sigma_k(r, h_2(r))\right\|\Big| \mathrm{d}r \\ &\leq K_*\Big(4 + \|h_1\|_{\mathscr{C}_m} + \|h_2\|_{\mathscr{C}_m}\Big) \int_0^u \left\|\sigma_k(r, h_1(r)) - \sigma_k(r, h_2(r))\right\| \mathrm{d}r \end{aligned}$$

for $h_1, h_2 \in \mathscr{C}_m$, continuity of functions $\sigma_k(r, \cdot)$ and the dominated convergence theorem.

Hence (5.20) yields

$$\begin{aligned} &\tilde{\mathbb{E}}_k\gamma(\varrho_s Y, \varrho_s B)\Big[\left\|M_k(t \wedge \tau_L(Y))\right\|^2 - \int_0^{t \wedge \tau_L(Y)} \left\|\sigma_k(r, Y(r))\right\|^2 \mathrm{d}r\Big] \\ &\qquad = \tilde{\mathbb{E}}_k\gamma(\varrho_s Y, \varrho_s B)\Big[\left\|M_k(s \wedge \tau_L(Y))\right\|^2 - \int_0^{s \wedge \tau_L(Y)} \left\|\sigma_k(r, Y(r))\right\|^2 \mathrm{d}r\Big]. \end{aligned}$$

Passing to the limit exactly in the same way as in the proof of Lemma 5.4.1 we obtain

$$\begin{aligned} &\tilde{\mathbb{E}}_0\gamma(\varrho_s Y, \varrho_s B)\Big[\left\|M_0(t \wedge \tau_L(Y))\right\|^2 - \int_0^{t \wedge \tau_L(Y)} \left\|\sigma_0(r, Y(r))\right\|^2 \mathrm{d}r\Big] \\ &\qquad = \tilde{\mathbb{E}}_0\gamma(\varrho_s Y, \varrho_s B)\Big[\left\|M_0(s \wedge \tau_L(Y))\right\|^2 - \int_0^{s \wedge \tau_L(Y)} \left\|\sigma_0(r, Y(r))\right\|^2 \mathrm{d}r\Big] \end{aligned}$$

provided that $L \in \mathbb{R}_+$ satisfies (5.17), and the proof may be completed easily. □

Proof of Lemma 5.4.4. Since μ_k and W are continuous local martingales, the process $\mu_k \otimes W - \langle\!\langle \mu_k, W \rangle\!\rangle$ is an $\mathbb{M}_{m \times n}$-valued local martingale. Let us denote $\mu_k = (\mu_k^i)_{i=1}^m$,

$W = (W^j)_{j=1}^n$ and $\sigma_k = (\sigma_k^{ij})_{i=1}^m{}_{j=1}^n$. Then

$$\begin{aligned}\langle \mu_k^i, W^j \rangle &= \left\langle \sum_{l=1}^n \int_0^\cdot \sigma_k^{il}(r, X_k(r))\, \mathrm{d}W^l(r), W^j \right\rangle \\ &= \sum_{l=1}^n \int_0^\cdot \sigma_k^{il}(r, X_k(r))\, \mathrm{d}\langle W^l, W^j \rangle_r \\ &= \int_0^\cdot \sigma_k^{ij}(r, X_k(r))\, \mathrm{d}r,\end{aligned}$$

therefore,

$$\mu_k \otimes W - \int_0^\cdot \sigma_k(r, X_k(r))\, \mathrm{d}r \tag{5.21}$$

is an $\mathbb{M}_{m\times n}$-valued local martingale. The process (5.21) stopped at $\tau_L(X_k, W)$ is bounded, hence it is a martingale and so

$$\begin{aligned}&\mathbb{E}\gamma(\varrho_s X_k, \varrho_s W)\Big[\big(\mu_k \otimes W\big)(t \wedge \tau_L(X_k, W)) - \int_0^{t\wedge\tau_L(X_k,W)} \sigma_k(r, X_k(r))\, \mathrm{d}r\Big] \\ &\quad = \mathbb{E}\gamma(\varrho_s X_k, \varrho_s W)\Big[\big(\mu_k \otimes W\big)(s \wedge \tau_L(X_k, W)) - \int_0^{s\wedge\tau_L(X_k,W)} \sigma_k(r, X_k(r))\, \mathrm{d}r\Big],\end{aligned}$$

whenever $0 \le s \le t \le T$ and γ is a continuous function as above. (Since $\mathscr{C}_m \times \mathscr{C}_n \cong \mathscr{C}_{m+n}$, it is clear how $\tau_L(f, g)$ is defined for $(f, g) \in \mathscr{C}_m \times \mathscr{C}_n$.) Now we may proceed as in the proof of Lemma 5.4.1. □

Proof of Theorem 5.1.1. Lemmas 5.4.1–5.4.4 having been established, it is straightforward to prove that $((U, \mathscr{U}, (\mathscr{U}_t), \tilde{\mathbb{P}}_0), B, Y)$ is a weak solution of (5.7). Since $\tilde{\mathbb{P}}_0 \circ Y(0)^{-1} = \tilde{\mathbb{P}}_k \circ Y(0)^{-1} = \mathbb{P} \circ \varphi^{-1} = \nu$ by our construction, it remains only to show that

$$Y(t) = Y(0) + \int_0^t b(r, Y(r))\, \mathrm{d}r + \int_0^t \sigma(r, Y(r))\, \mathrm{d}B(r)$$

for any $t \in [0, T]$ $\tilde{\mathbb{P}}_0$-almost surely, that is

$$M_0(t) = \int_0^t \sigma(r, Y(r))\, \mathrm{d}B(r) \quad \text{for all } t \in [0, T]\ \tilde{\mathbb{P}}_0\text{-almost surely.} \tag{5.22}$$

Obviously, (5.22) is equivalent to

$$\left\langle M_0 - \int_0^\cdot \sigma(r, Y(r))\, \mathrm{d}B(r) \right\rangle_T = 0 \quad \tilde{\mathbb{P}}_0\text{-almost surely.} \tag{5.23}$$

We have

$$\begin{aligned}\Big\langle M_0 - \int_0^\cdot \sigma(r, Y(r))\,\mathrm{d}B(r)\Big\rangle_T &= \langle M_0\rangle_T + \Big\langle \int_0^\cdot \sigma(r, Y(r))\,\mathrm{d}B(r)\Big\rangle_T \\ &\quad - 2\sum_{i=1}^m \Big\langle M_0^i, \sum_{j=1}^n \int_0^\cdot \sigma^{ij}(r, Y(r))\,\mathrm{d}B^j(r)\Big\rangle_T \\ &= \langle M_0\rangle_T + \int_0^T \|\sigma(r, Y(r))\|^2\,\mathrm{d}r \\ &\quad - 2\sum_{i=1}^m \Big\langle M_0^i, \sum_{j=1}^n \int_0^\cdot \sigma^{ij}(r, Y(r))\,\mathrm{d}B^j(r)\Big\rangle_T.\end{aligned}$$

By Lemma 5.4.3,

$$\langle M_0\rangle_T = \int_0^T \|\sigma(r, X(r))\|^2\,\mathrm{d}r,$$

and by Lemma 5.4.4 we obtain

$$\begin{aligned}\sum_{i=1}^m\sum_{j=1}^n \Big\langle M_0^i, \int_0^\cdot \sigma^{ij}(r, Y(r))\,\mathrm{d}B^j(r)\Big\rangle_T &= \sum_{i=1}^m\sum_{j=1}^n \int_0^T \sigma^{ij}(r, Y(r))\,\mathrm{d}\langle M_0^i, B^j\rangle_r \\ &= \sum_{i=1}^m\sum_{j=1}^n \int_0^T \big(\sigma^{ij}(r, Y(r))\big)^2\,\mathrm{d}r \\ &= \int_0^T \|\sigma(r, Y(r))\|^2\,\mathrm{d}r,\end{aligned}$$

hence (5.23) holds true. □

Remark 5.4.8. If the coefficients b and σ of the equation (5.7) are defined on $\mathbb{R}_+ \times \mathbb{R}^m$ and satisfy the assumptions of Theorem 5.1.1 there, then there exists a weak solution to (5.7) defined for all times $t \geq 0$. The proof remains almost the same, only its part concerning tightness requires small modifications. However, it suffices to realize that the space $\mathscr{C}(\mathbb{R}_+; \mathbb{R}^V)$ equipped with the topology of locally uniform convergence is a Polish space whose Borel σ-algebra is generated by the projections $f \mapsto f(t)$, $t \geq 0$ and whose closed subset K is compact if and only if $\{\varrho_T f;\ f \in K\}$ is a compact subset of $\mathscr{C}([0,T]; \mathbb{R}^V)$ for all $T \geq 0$.

Remark 5.4.9. Tracing the proofs in Section 5.4, we can check easily that, unlike the proof of tightness in Section 5.3, they depend only on the following properties of the coefficients $b = b_0$, $\sigma = \sigma_0$ and their approximations b_k, σ_k:

(i) the functions $b_k(r, \cdot)$, $\sigma_k(r, \cdot)$ are continuous on $\mathbb{R}^m$ for any $r \in [0,T]$ and $k \geq 0$,

(ii) $b_k(r, \cdot) \to b(r, \cdot)$, $\sigma_k(r, \cdot) \to \sigma(r, \cdot)$ locally uniformly on $\mathbb{R}^m$ as $k \to \infty$ for any $r \in [0,T]$,

(iii) the functions b_k, σ_k are locally bounded uniformly in $k \geq 0$, i.e.

$$\sup_{k\geq 0}\ \sup_{r\in[0,T]}\ \sup_{\|z\|\leq L} \big\{\|b_k(r,z)\| \vee \|\sigma_k(r,z)\|\big\} < \infty$$

for each $L \geq 0$.

As a consequence, Theorem 5.1.1 remains valid if existence of a suitable Lyapunov function is supposed instead of the linear growth hypothesis. One proceeds as in the proof of Theorem 5.1.1, approximating the coefficients b and σ by bounded continuous functions that satisfy the same Lyapunov estimate as b and σ. However, the proof of tightness is more technical, although no fundamentally new ideas are needed; details may be found in a companion paper [35].

5.5 On Weak Solutions to SDEs II.

Let us consider a stochastic differential equation

$$\mathrm{d}X = b(t,X)\,\mathrm{d}t + \sigma(t,X)\,\mathrm{d}W, \quad X(0) \overset{d}{\sim} \nu, \tag{5.24}$$

where $b\colon [0,T]\times\mathbb{R}^m \to \mathbb{R}^m$, $\sigma\colon [0,T]\times\mathbb{R}^m \to \mathbb{M}_{m\times n}$ are Borel functions and ν is a Borel probability measure on $\mathbb{R}^m$. (In what follows, we shall denote by $\mathbb{M}_{m\times n}$ the space of all m-by-n matrices over $\mathbb{R}$ endowed with the Hilbert-Schmidt norm $\|A\| = (\operatorname{Tr} AA^*)^{1/2}$.)

If the coefficients b and σ are continuous in the second variable and satisfy a linear growth hypothesis

$$\sup_{t\in[0,T]}\ \sup_{x\in\mathbb{R}^m} \frac{\|b(t,x)\| + \|\sigma(t,x)\|}{1+\|x\|} < \infty, \tag{5.25}$$

then there exists a weak solution to (5.24) by a theorem established by A. V. Skorokhod some fifty years ago. All proofs of his result that we know have a common basic structure: (5.24) is approximated with equations having a solution, then tightness of laws of solutions to these approximating equations is shown and finally cluster points of the set of laws are identified as weak solutions to (5.24). In the first part of our paper [34] we proposed a new, fairly elementary, version of this argument. In [34] tightness is proved by means of compactness properties of fractional integrals, while the identification procedure uses results on preservation of the local martingale property under convergence in law, avoiding thus both Skorokhod's theorem on almost surely converging realizations of converging laws and results on integral representation of martingales with absolutely continuous quadratic variation, see [34] for more details and references.

The purpose of the present paper, which may be viewed as a short addendum to [34], is to show that the new method may be used even if (5.25) is relaxed to existence of a suitable Lyapunov function. Namely, we shall prove the following result.

Theorem 5.5.1. *Assume that a hypothesis*

(A) *$b(r,\cdot)$ and $\sigma(r,\cdot)$ are continuous on $\mathbb{R}^m$ for any $r\in[0,T]$ and both functions b, σ are locally bounded on $[0,T]\times\mathbb{R}^m$, i.e.*

$$\sup_{r\in[0,T]}\ \sup_{\|z\|\le L} \bigl\{\|b(r,z)\| \vee \|\sigma(r,z)\|\bigr\} < \infty$$

for all $L\ge 0$,

is satisfied and a function $V\in\mathscr{C}^2(\mathbb{R}^m)$ may be found such that

(L1) *there exists an increasing function $\kappa\colon \mathbb{R}_+ \to\,]0,\infty[$ such that*

$$\lim_{r\to\infty} \kappa(r) = +\infty$$

and $V(x) \ge \kappa(\|x\|)$ for all $x\in\mathbb{R}^m$, (L2) *there exists* $\gamma \geq 0$ *such that*

$$\big\langle b(t,x), DV(x)\big\rangle + \frac{1}{2}\operatorname{Tr}\big(\sigma(t,x)^* D^2V(x)\sigma(t,x)\big) \leq \gamma V(x)$$

for all $(t,x) \in [0,T] \times \mathbb{R}^m$.

Then there exists a weak solution to (5.24).

(By DV and D^2V we denote the first and second Fréchet derivative of V, respectively.) The assumption (L2) is the well known Khas'minskii's condition for non-explosion (see [43], Theorem 3.5, where equations with locally Lipschitz continuous coefficients are considered), however, we do not work with local solutions and construct global solutions directly. To prove Theorem 5.5.1 we approximate coefficients b and σ with bounded continuous functions. Essentially, we mimick the proof of tightness of the laws of solutions to approximating equations from [34], however, in absence of (5.25) we do not have uniform moment estimates for approximating processes X_k at our disposal, instead, we have to resort to a well known trick from stability theory and show, roughly speaking, that $(e^{-\gamma t}V(X_k(t)))$ are supermartingales. As a consequence, the proof is less straightforward than the corresponding one in [34]. Once tightness is proved, the identification procedure from [34] may be applied without any change, since it does not depend on any particular form of approximations. More precisely, in [34], Remark 3.2, we proved:

Proposition 5.5.2. *Let the assumption* (A) *be satisfied. Let there exist Borel functions* $b_k\colon [0,T]\times\mathbb{R}^m \to \mathbb{R}^m$ *and* $\sigma_k\colon [0,T]\times\mathbb{R}^m \to \mathbb{M}_{m\times n}$, $k \geq 1$, *such that*

1° $b_k(r,\cdot)$, $\sigma_k(r,\cdot)$ *are continuous on* $\mathbb{R}^m$ *for any* $r\in[0,T]$ *and* $k \geq 1$,

2° $b_k(r,\cdot) \to b(r,\cdot)$, $\sigma_k(r,\cdot) \to \sigma(r,\cdot)$ *locally uniformly on* $\mathbb{R}^m$ *as* $k\to\infty$ *for any* $r\in[0,T]$,

3° *the functions* b_k, σ_k *are locally bounded on* $[0,T]\times\mathbb{R}^m$ *uniformly in* $k\geq 1$, *that is*

$$\sup_{k\geq 1}\ \sup_{r\in[0,T]}\ \sup_{\|z\|\leq L} \big\{\|b_k(r,z)\| \vee \|\sigma_k(r,z)\|\big\} < \infty$$

for each $L \geq 1$.

Suppose that for any $k \geq 1$ *there exists a weak solution* $((\Omega_k, \mathscr{F}^k, (\mathscr{F}^k_t), \mathbb{P}_k), W_k, X_k)$ *to the problem*

$$\mathrm{d}X = b_k(t,X)\,\mathrm{d}t + \sigma_k(t,X)\,\mathrm{d}W, \quad X(0) \overset{d}{\sim} \nu. \tag{5.26}$$

If $\{\mathbb{P}_k \circ X_k^{-1};\ k\geq 1\}$ *is a tight set of probability measures on* $\mathscr{C}([0,T];\mathbb{R}^m)$ *then there exists a weak solution to* (5.24).

Before proceeding to the proof of Theorem 5.5.1, we shall recall some definitions and give a few illustrative examples. First, a weak solution to (5.24) is a triple $((G,\mathscr{G},(\mathscr{G}_t),\mathbb{Q}),W,X)$, where $(G,\mathscr{G},(\mathscr{G}_t),\mathbb{Q})$ is a stochastic basis with a filtration $(\mathscr{G}_t)$ that satisfies the usual conditions, W is an n-dimensional $(\mathscr{G}_t)$-Wiener process and X is an $\mathbb{R}^m$-valued $(\mathscr{G}_t)$-progressively measurable process such that $\mathbb{Q}\circ X(0)^{-1} = \nu$ and

$$X(t) = X(0) + \int_0^t b(r,X(r))\,\mathrm{d}r + \int_0^t \sigma(r,X(r))\,\mathrm{d}W(r)$$

for all $t \in [0,T]$ $\mathbb{Q}$-almost surely. In the proof we use the Riemann-Liouville (or fractional integral) operator: if $q \in]1,\infty]$, $\alpha \in]\frac{1}{q},1]$ and $f \in L^q([0,T];\mathbb{R}^m)$, a function $R_\alpha f : [0,T] \to \mathbb{R}^m$ is defined by

$$\big(R_\alpha f\big)(t) = \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad 0 \le t \le T.$$

The (easy) properties of $R_\alpha\colon f \mapsto R_\alpha f$ which we need are summarized in [34], Lemma 2.2. Finally, by $\mathscr{C}_{1,2}$ we shall denote the set of all $h \in \mathscr{C}^1([0,T]\times\mathbb{R}^m)$ such that $h(t,\cdot) \in \mathscr{C}^2(\mathbb{R}^m)$ for each $t \in [0,T]$ and D_h, $D_x^2 h$ are continuous functions on $[0,T]\times\mathbb{R}^m$, $D_x h(t,x)$ and $D_x^2 h(t,x)$ being the first and second Fréchet derivative of $h(t,\cdot)$ at the point x, respectively.

Example 5.5.3. If the coefficients b and σ satisfy (A) and (5.25) then Theorem 5.5.1 is applicable. More generally, assume that

$$2\langle b(t,x),x\rangle + \|\sigma(t,x)\|^2 \le K\big(1+\|x\|^2\big)$$

for some $K < \infty$ and all $t \in [0,T]$, $x \in \mathbb{R}^m$. Then the Lyapunov function $V\colon x \mapsto 1+\|x\|^2$ satisfies (L1) and (L2).

Example 5.5.4. Suppose that $\sigma\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ is a function bounded on bounded sets and $\sigma(t,\cdot) \in \mathscr{C}(\mathbb{R})$ for each $t \in [0,T]$. Then we may use Theorem 5.5.1 with a Lyapunov function $V\colon x \mapsto \log(e+x^2)$ to deduce that a stochastic differential equation

$$\mathrm{d}X = \sigma(t,X)\,\mathrm{d}W, \quad X_0 \overset{\mathscr{D}}{\rightarrow}\!\sim \nu$$

has a weak solution. Of course, it is known that explosions cannot occur for one-dimensional stochastic differential equations without drift, irrespective of growth and continuity properties of σ, but a proof based on Lyapunov functions, when available, is much simpler than the one in the general case.

Example 5.5.5. Let us consider a stochastic nonlinear oscillator $\ddot{x} + x^{2k+1} = \sigma(x)\dot{w}$, where $k \in \mathbb{N}$ and $\sigma \in \mathscr{C}(\mathbb{R})$, that is rigorously, a system

$$\mathrm{d}X = Y\,\mathrm{d}t, \quad \mathrm{d}Y = -X^{2k+1}\,\mathrm{d}t + \sigma(X)\,\mathrm{d}W. \tag{5.27}$$

Theorem 5.5.1 with a choice

$$V\colon \mathbb{R}^2 \longrightarrow \mathbb{R}, \ \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \log\Big(e + \frac{x^{2k+2}}{2k+2} + \frac{y^2}{2}\Big)$$

implies that there exists a weak solution of (5.27) with an arbitrary initial condition ν provided $\sigma^2(x) = O(x^{2k+2})$, $x \to \pm\infty$.

Proof of Theorem 5.5.1. For $k \ge 1$, let us define

$$b_k(t,x) = \begin{cases} b(t,x), & 0 \le t \le T,\ \|x\| \le k, \\ b(t,x)\big(2-k^{-1}\|x\|\big)^2, & 0 \le t \le T,\ k < \|x\| \le 2k, \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$\sigma_k(t,x) = \begin{cases} \sigma(t,x), & 0 \le t \le T,\ \|x\| \le k, \\ \sigma(t,x)\big(2-k^{-1}\|x\|\big), & 0 \le t \le T,\ k < \|x\| \le 2k, \\ 0 & \text{elsewhere.} \end{cases}$$

Obviously, hypotheses 1° and 2° of Proposition 5.5.2 are satisfied, moreover $\|b_k\| \leq \|b\|$ and $\|\sigma_k\| \leq \|\sigma\|$ on $[0,T] \times \mathbb{R}^m$ for all $k \geq 1$ and thus 3° is satisfied as well. The coefficients b_k and σ_k are bounded, so Theorem 0.1 from [34] implies that there exists a weak solution $((\Omega_k, \mathscr{F}^k, (\mathscr{F}_t^k), \mathbb{P}_k), W_k, X_k)$ of (5.26). Therefore, Theorem 5.5.1 will follow from Proposition 5.5.2 provided we show that $\{\mathbb{P}_k \circ X_k^{-1};\ k \geq 1\}$ is a tight set of measures.

Towards this end, let us define for any $h \in \mathscr{C}_{1,2}$ and $k \geq 1$ a function $L_k h\colon [0,T] \times \mathbb{R}^m \to \mathbb{R}$ by

$$(L_k h)(t,x) = \big\langle b_k(t,x), D_x h(t,x)\big\rangle + \frac{1}{2}\operatorname{Tr}\big(\sigma_k(t,x)^* D_x^2 h(t,x)\sigma_k(t,x)\big),$$

$(t,x) \in [0,T] \times \mathbb{R}^m$. The definition of b_k and σ_k and the assumption (L2) imply that

$$L_k V(t,x) \leq \gamma V(x) \quad \text{for all } k \geq 1 \text{ and } (t,x) \in [0,T] \times \mathbb{R}^m.$$

A straightforward calculation shows that if we set $U(t,x) = e^{-\gamma t} V(x)$ then

$$\Big(\frac{\partial U}{\partial t} + L_k U\Big)(t,x) \leq 0 \quad \text{for all } k \geq 1 \text{ and } (t,x) \in [0,T] \times \mathbb{R}^m. \tag{5.28}$$

Let us fix $k \geq 1$ for a while. From the Itô formula we get

$$\begin{aligned} &U(t\wedge\varrho, X_k(t\wedge\varrho)) - U(s\wedge\varrho, X_k(s\wedge\varrho)) \\ &\quad= \int_{s\wedge\varrho}^{t\wedge\varrho} \Big(\frac{\partial U}{\partial t} + L_k U\Big)(r, X_k(r))\,\mathrm{d}r + \int_{s\wedge\varrho}^{t\wedge\varrho} D_x U(r, X_k(r))^* \sigma_k(r, X_k(r))\,\mathrm{d}W_k(r), \end{aligned}$$

and thus

$$\begin{aligned} &U(t\wedge\varrho, X_k(t\wedge\varrho)) - U(s\wedge\varrho, X_k(s\wedge\varrho)) \\ &\qquad\qquad \leq \int_{s\wedge\varrho}^{t\wedge\varrho} D_x U(r, X_k(r))^* \sigma_k(r, X_k(r))\,\mathrm{d}W_k(r) \end{aligned} \tag{5.29}$$

by (5.28), whenever $s,t \in [0,T]$, $s \leq t$ and ϱ is an $[0,T]$-valued $(\mathscr{F}_r^k)$-stopping time.

First, let us choose $s = 0$, $L \geq 0$, and

$$\varrho = \tau_L \equiv \inf\big\{r \geq 0;\ \|X_k(r)\| \geq L\big\}$$

(where we set $\inf\emptyset = T$). Since $U(0,\cdot) = V$ we obtain

$$U(t\wedge\tau_L, X_k(t\wedge\tau_L)) \leq V(X_k(0)) + \int_0^{t\wedge\tau_L} D_x U(r, X_k(r))^* \sigma_k(r, X_k(r))\,\mathrm{d}W_k(r).$$

Let $\chi \subseteq \mathbb{R}^m$ be an arbitrary Borel set such that

$$\int_\chi V(z)\,\mathrm{d}\nu(z) < \infty. \tag{5.30}$$

(Plainly, any compact set χ satisfies (5.30).) Denoting by A the set $\{X_k(0) \in \chi\} \in \mathscr{F}_0^k$ we get

$$\begin{aligned}&\mathbf{1}_A U(t\wedge\tau_L, X_k(t\wedge\tau_L))\\&\qquad\qquad\le \mathbf{1}_A V(X_k(0)) + \int_0^{t\wedge\tau_L} \mathbf{1}_A D_x U(r, X_k(r))^* \sigma_k(r, X_k(r))\,\mathrm{d}W_k(r).\end{aligned}$$

As $\mathbf{1}_A \mathbf{1}_{[0,\tau_L[}(\cdot) D_x U(\cdot, X_k(\cdot))^* \sigma_k(\cdot, X_k(\cdot))$ is bounded on $[0,T]\times\Omega_k$ due to continuity of $D_x U$, local boundedness of σ_k and the definition of τ_L, we have

$$\begin{aligned}\mathbb{E}_k \mathbf{1}_A U(t\wedge\tau_L, X_k(t\wedge\tau_L)) &\le \mathbb{E}_k \mathbf{1}_A V(X_k(0)) = \mathbb{E}_k \mathbf{1}_\chi(X_k(0)) V(X_k(0))\\&= \int_\chi V(z)\,\mathrm{d}\nu(z);\end{aligned}$$

the right-hand side is independent of $L \ge 0$. Clearly, $\{\tau_L = T\} \uparrow \Omega_k$ $\mathbb{P}_k$-almost surely as $L \to \infty$, since X_k has continuous trajectories, so

$$\mathbb{E}_k \mathbf{1}_A U(t, X_k(t)) \le \int_\chi V(z)\,\mathrm{d}\nu(z) < \infty$$

by the Fatou lemma.

In particular, if $s,t \in [0,T]$, $s \le t$, then the conditional expectation

$$\mathbb{E}_k\big(\mathbf{1}_A U(t, X_k(t)) \,\big|\, \mathscr{F}_s^k\big)$$

is well defined. Using (5.29) with the stopping time τ_L, replacing the Fatou lemma with its version for conditional expectations but otherwise proceeding as above we arrive at an estimate

$$\mathbb{E}_k\big(\mathbf{1}_A U(t, X_k(t)) \,\big|\, \mathscr{F}_s^k\big) \le \mathbf{1}_A U(s, X_k(s)), \quad 0 \le s \le t \le T.$$

Consequently, $\big(\mathbf{1}_A U(t, X_k(t)),\ 0 \le t \le T\big)$ is a nonnegative continuous supermartingale. The maximal inequality for supermartingales implies

$$\begin{aligned}\mathbb{P}_k\Big\{\sup_{0\le t\le T} \mathbf{1}_\chi(X_k(0)) U(t, X_k(t)) > \lambda\Big\} &\le \frac{1}{\lambda}\mathbb{E}_k \mathbf{1}_\chi(X_k(0)) V(X_k(0))\\&= \frac{1}{\lambda}\int_\chi V(z)\,\mathrm{d}\nu(z),\end{aligned}$$

hence, by the definition of U,

$$\mathbb{P}_k\Big\{\sup_{0\le t\le T} \mathbf{1}_\chi(X_k(0)) V(X_k(t)) > \lambda\Big\} \le \frac{e^{\gamma T}}{\lambda}\int_\chi V\,\mathrm{d}\nu$$

for all $\lambda > 0$; the estimate is uniform in $k \ge 1$. From the assumption (L1) we deduce that

$$\mathbb{P}_k\Big\{\sup_{0\le t\le T} \mathbf{1}_\chi(X_k(0)) \|X_k(t)\| > \lambda\Big\} \le \frac{e^{\gamma T}}{\kappa(\lambda)}\int_\chi V\,\mathrm{d}\nu \tag{5.31}$$

holds for all $\lambda > 0$ and $k \ge 1$.

Now the proof of tightness of $\{\mathbb{P}_k \circ X_k^{-1};\ k \ge 1\}$ can be completed essentially in the same manner as in the proof of Proposition 2.1 in [34]. Let an arbitrary $\varepsilon > 0$ be given, we want to find a relatively compact set $K \subseteq \mathscr{C}([0,T];\mathbb{R}^m)$ so that

$$\sup_{k\geq 1} \mathbb{P}_k\big\{X_k \notin K\big\} \leq \varepsilon. \tag{5.32}$$

Let us take an arbitrary $p \in\,]2,\infty[$ and $\alpha \in\,]\frac{1}{p},\frac{1}{2}[$ and recall that X_k has a representation (see e.g. [34], Lemma 2.5)

$$X_k(t) = X_k(0) + \big[R_1 b_k(\cdot, X_k(\cdot))\big](t) + \frac{\sin \pi\alpha}{\pi}(R_\alpha Z_k)(t), \quad 0 \leq t \leq T,$$

where

$$Z_k(t) = \int_0^t (t-s)^{-\alpha}\sigma_k(s, X_k(s))\,\mathrm{d}W_k(s), \quad 0 \leq t \leq T.$$

The process Z_k is plainly well defined for every $t \in [0,T]$, since σ_k is a bounded function. Let $H \subseteq \mathbb{R}^m$ be a compact set such that $\nu(\mathbb{R}^m \setminus H) = \mathbb{P}_k\{X_k(0) \notin H\} < \varepsilon/8$. The set

$$K = \Big\{f \in \mathscr{C}([0,T];\mathbb{R}^m);\ f = x + R_1 v + \frac{\sin\pi\alpha}{\pi}R_\alpha w,\ x \in H,$$
$$v, w \in L^p(0,T;\mathbb{R}^m),\ |v|_p \vee |w|_p \leq \Lambda\Big\},$$

where by $|\cdot|_p$ the norm of $L^p(0,T;\mathbb{R}^m)$ is denoted, is relatively compact owing to compactness of the operators R_1 and R_α. It remains to show that $\Lambda > 0$ may be found for K to satisfy (5.32).

From (5.31) and (L1) we obtain that there exists $\lambda_0 > 0$ such that

$$\sup_{k\geq 1}\mathbb{P}_k\Big\{\mathbf{1}_H(X_k(0)) \sup_{0\leq t\leq T}\|X_k(t)\| > \lambda_0\Big\} \leq \frac{e^{\gamma T}}{\kappa(\lambda_0)}\int_H V\,\mathrm{d}\nu < \frac{\varepsilon}{8},$$

therefore the choice of H gives

$$\sup_{k\geq 1}\mathbb{P}_k\Big\{\sup_{0\leq t\leq T}\|X_k(t)\| > \lambda_0\Big\} < \frac{\varepsilon}{4}.$$

Hence if we set

$$B_k = \Big\{\omega \in \Omega_k;\ \sup_{0\leq t\leq T}\|X_k(t,\omega)\| \leq \lambda_0\Big\},$$

then $\mathbb{P}_k(\Omega_k \setminus B_k) < \varepsilon/4$ for all $k \geq 1$.

Obviously,

$$\mathbb{P}_k\{X_k \notin K\} \leq \mathbb{P}_k\{X_k(0) \notin H\} + \mathbb{P}_k\big\{|b_k(\cdot, X_k(\cdot))|_p > \Lambda\big\} + \mathbb{P}_k\big\{|Z_k|_p > \Lambda\big\}.$$

By the Chebyshev inequality, we get

$$\begin{aligned}
\mathbb{P}_k\big\{|b_k(\cdot, X_k(\cdot))|_p > \Lambda\big\} &\leq \mathbb{P}_k(\Omega_k \setminus B_k) + \mathbb{P}_k\big\{\omega \in B_k;\ |b_k(\cdot, X_k(\cdot))|_p > \Lambda\big\} \\
&\leq \frac{\varepsilon}{4} + \frac{1}{\Lambda^p}\mathbb{E}_k\mathbf{1}_{B_k}\int_0^T \|b_k(r, X_k(r))\|^p\,\mathrm{d}r \\
&\leq \frac{\varepsilon}{4} + \frac{T}{\Lambda^p}\sup_{\substack{0\leq t\leq T\\ \|z\|\leq\lambda_0}}\|b_k(t,z)\|^p \\
&\leq \frac{\varepsilon}{4} + \frac{T}{\Lambda^p}\sup_{\substack{0\leq t\leq T\\ \|z\|\leq\lambda_0}}\|b(t,z)\|^p.
\end{aligned}$$

The right-hand side is independent of $k \geq 1$, so there exists $\Lambda_1 > 0$ such that

$$\sup_{k\geq 1} \mathbb{P}_k\bigl\{|b_k(\cdot, X_k(\cdot))|_p > \Lambda\bigr\} \leq \frac{\varepsilon}{3}$$

for all $\Lambda \geq \Lambda_1$. The norm $|Z_k|_p$ may be estimated analogously. Clearly,

$$\begin{aligned}
\mathbb{P}_k\{|Z_k|_p > \Lambda\} &\leq \mathbb{P}_k(\Omega_k \setminus B_k) + \mathbb{P}_k\{\omega \in B_k;\ |Z_k|_p > \Lambda\} \\
&\leq \frac{\varepsilon}{4} + \mathbb{P}_k\{\omega \in B_k;\ |Z_k|_p > \Lambda\}.
\end{aligned}$$

For each $k \geq 1$ let us define an $(\mathscr{F}_t^k)$-stopping time ζ_k by

$$\zeta_k = \inf\{t \in [0,T];\ \|X_k(t)\| > \lambda_0\},$$

setting again $\inf \emptyset = T$. Using the Chebyshev and Young inequalities and noting that $\zeta_k = T$ on B_k we obtain

$$\begin{aligned}
&\mathbb{P}_k\{\omega \in B_k;\ |Z_k|_p > \Lambda\} \\
&\qquad \leq \frac{1}{\Lambda^p}\mathbb{E}_k\mathbf{1}_{B_k}\int_0^T \|Z_k(s)\|^p \,\mathrm{d}s \\
&\qquad = \frac{1}{\Lambda^p}\mathbb{E}_k\mathbf{1}_{B_k}\int_0^T \left\|\int_0^s (s-u)^{-\alpha}\sigma_k(u, X_k(u))\,\mathrm{d}W(u)\right\|^p \mathrm{d}s \\
&\qquad = \frac{1}{\Lambda^p}\mathbb{E}_k\mathbf{1}_{B_k}\int_0^T \left\|\int_0^s (s-u)^{-\alpha}\mathbf{1}_{[0,\zeta_k[}(u)\sigma_k(u, X_k(u))\,\mathrm{d}W(u)\right\|^p \mathrm{d}s \\
&\qquad \leq \frac{1}{\Lambda^p}\mathbb{E}_k\int_0^T \left\|\int_0^s (s-u)^{-\alpha}\mathbf{1}_{[0,\zeta_k[}(u)\sigma_k(u, X_k(u))\,\mathrm{d}W(u)\right\|^p \mathrm{d}s \\
&\qquad \leq \frac{C_p}{\Lambda^p}\mathbb{E}_k\int_0^T \left(\int_0^s (s-u)^{-2\alpha}\mathbf{1}_{[0,\zeta_k[}(u)\|\sigma_k(u, X_k(u))\|^2\,\mathrm{d}u\right)^{p/2} \mathrm{d}s \\
&\qquad \leq \frac{C_p}{\Lambda^p}\left(\int_0^T u^{-2\alpha}\,\mathrm{d}u\right)^{p/2}\mathbb{E}_k\int_0^T \mathbf{1}_{[0,\zeta_k[}(u)\|\sigma_k(u, X_k(u))\|^p\,\mathrm{d}u \\
&\qquad \leq \frac{C_pT}{\Lambda^p}\left(\int_0^T u^{-2\alpha}\,\mathrm{d}u\right)^{p/2}\sup_{\substack{0\leq t\leq T\\ \|z\|\leq\lambda_0}}\|\sigma_k(t,x)\|^p \\
&\qquad \leq \frac{C_pT}{\Lambda^p}\left(\int_0^T u^{-2\alpha}\,\mathrm{d}u\right)^{p/2}\sup_{\substack{0\leq t\leq T\\ \|z\|\leq\lambda_0}}\|\sigma(t,x)\|^p,
\end{aligned}$$

where C_p is a constant coming from the Burkholder-Gundy-Davis inequality. We see that there exists a constant $\Lambda_2 > 0$ such that

$$\sup_{k\geq 1}\mathbb{P}_k\bigl\{|Z_k|_p \geq \Lambda\bigr\} < \frac{\varepsilon}{3}$$

for all $\Lambda \geq \Lambda_2$ and hence the proof may be completed easily. □

5.A Appendix

To keep the paper self-contained as much as possible, we provide here proofs of Lemmas 5.4.5 and 5.4.6.

Proof of Lemma 5.4.5. Choose $f \in \mathscr{C}_V$ and $L > 0$ arbitrarily. The function $K \mapsto \tau_K(f)$ is obviously nondecreasing, hence it has a left-hand limit at the point L and

$$\lim_{K \to L-} \tau_K(f) \leq \tau_L(f). \tag{5.33}$$

If $\|f\|_{\mathscr{C}_V} < L$ then $\|f\|_{\mathscr{C}_V} < L - \delta$ for some $\delta > 0$ and thus $\tau_L(f) = T = \tau_K(f)$ for all $K \in [L-\delta, L]$, so we may assume that $\|f\|_{\mathscr{C}_V} \geq L$. Then $\|f(\tau_K(f))\| \geq K$ for all $K \in [0, L]$ and continuity of f yields

$$\left\|f\big(\lim_{K \to L-} \tau_K(f)\big)\right\| = \lim_{K \to L-} \left\|f(\tau_K(f))\right\| \geq \lim_{K \to L-} K = L,$$

whence

$$\tau_L(f) \leq \lim_{K \to L-} \tau_K(f),$$

which together with (5.33) proves the statement (i).

To prove (ii), take an arbitrary sequence $\{f_r\}$ in $\mathscr{C}_V$ such that $f_r \to f$ uniformly on $[0, T]$ as $r \to \infty$. Let $\varepsilon > 0$, then

$$\max_{[0, \tau_L(f) - \varepsilon]} \|f\| < L,$$

so there exists $r_0 \in \mathbb{N}$ such that

$$\max_{[0, \tau_L(f) - \varepsilon]} \|f_r\| < L$$

for all $r \geq r_0$, thus $\tau_L(f_r) \geq \tau_L(f) - \varepsilon$ for all $r \geq r_0$. Since ε was arbitrary,

$$\liminf_{r \to \infty} \tau_L(f_r) \geq \tau_L(f),$$

that is, τ_L is lower semicontinuous at the point f.

Finally, assume in addition that $\tau_\bullet(f)$ is continuous at the point L. If $\tau_L(f) = T$ then

$$T = \tau_L(f) \leq \liminf_{r \to \infty} \tau_L(f_r) \leq \limsup_{r \to \infty} \tau_L(f_r) \leq T$$

(note that τ_L is $[0, T]$-valued) and we are done. So assume that $\tau_L(f) < T$ and take an arbitrary $\varepsilon > 0$ satisfying $\tau_L(f) + \varepsilon < T$. By continuity, a $K > L$ may be found such that $\tau_K(f) < \tau_L(f) + \varepsilon$. Consequently,

$$\max_{[0, \tau_L(f) + \varepsilon]} \|f\| \geq K > L,$$

thus

$$\max_{[0, \tau_L(f) + \varepsilon]} \|f_r\| \geq L$$

for all r sufficiently large, that is $\tau_L(f_r) \leq \tau_L(f) + \varepsilon$ for all r sufficiently large, which implies

$$\limsup_{r \to \infty} \tau_L(f_r) \leq \tau_L(f)$$

and τ_L is upper semicontinuous at f. □

Proof of Lemma 5.4.6. Here we follow the book [41] closely. First, note that for any given $u > 0$ $\boldsymbol{q}$-almost any trajectory of $\tau_\bullet(Z)$ has only finitely many jumps of size greater than u. For brevity, set

$$\Delta\tau_L(Z) = \lim_{M\to L+} \tau_M(Z) - \tau_L(Z)$$

and define recursively random times

$$\Sigma_0(u) = 0,\ \Sigma_p(u) = \inf\bigl\{L > \Sigma_p(u);\ \Delta\tau_L(Z) > u\bigr\}, \quad u > 0,\ p \in \mathbb{N}.$$

Plainly, the set

$$\bigl\{L \geq 0;\ \boldsymbol{q}\{\Sigma_p(u) = L\} > 0\bigr\}$$

is at most countable for any $p \in \mathbb{N}$ and $u > 0$, hence it only remains to note that

$$\bigl\{L \geq 0;\ \boldsymbol{q}\{\Delta\tau_L(Z) > 0\} > 0\bigr\} = \bigcup_{p=0}^{\infty}\bigcup_{r=1}^{\infty}\bigl\{L \geq 0;\ \boldsymbol{q}\{\Sigma_p(r^{-1}) = L\} > 0\bigr\}.$$

$\square$

Bibliography

[1] H. Amann, M. Hieber, G. Simonett, *Bounded H_∞-calculus for elliptic operators*, Diff. Int. Eq. **7** (1994) 613–653.

[2] H. Amann, *Compact embeddings of vector-valued Sobolev and Besov spaces*, Glass. Mat., III. Ser. **35** (55) (2000) 161–177.

[3] H. Amann, *Linear and Quasilinear Parabolic Problems,* Birkhäuser Verlag, Basel, Boston, Berlin, 1995.

[4] L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[5] M. T. Barlow, *One dimensional stochastic differential equation with no strong solution*, J. London Math. Soc. (2) **26** (2) (1982) 335–347.

[6] C. Bauzet, G. Vallet, P. Wittbolt, *The Cauchy problem for conservation laws with a multiplicative noise*, Journal of Hyp. Diff. Eq. **9** (4) (2012) 661–709.

[7] F. Berthelin, J. Vovelle, *A BGK approximation to scalar conservation laws with discontinuous flux*, Proc. of the Royal Society of Edinburgh A **140** (5) (2010) 953–972.

[8] N. Bourbaki, *Integration II*, Springer, Berlin, 2004.

[9] Z. Brzeźniak, *On stochastic convolution in Banach spaces and applications*, Stoch. Stoch. Rep. **61** (1997) 245–295.

[10] Z. Brzeźniak, *Stochastic partial differential equations in M-type 2 Banach spaces*, Potential Anal. **4** (1995) 1–45.

[11] Z. Brzeźniak, M. Ondreját, *Strong solutions to stochastic wave equations with values in Riemannian manifolds*, Journal of Functional Analysis **253** (2) (2007) 449–481.

[12] J. Carrillo, *Entropy solutions for nonlinear degenerate problems*, Arch. Rational Mech. Anal. **147** (1999) 269–361.

[13] G. Q. Chen, B. Perthame, *Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **20** (4) (2003) 645–668.

[14] D. L. Cohn, *Measure Theory*, Birkhäuser, Boston, Basel, Stuttgart, 1980.

[15] G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Encyclopedia Math. Appl., vol. 44, Cambridge University Press, Cambridge, 1992.

[16] A. Debussche, J. Vovelle, *Scalar conservation laws with stochastic forcing*, J. Funct. Anal. **259** (2010) 1014–1042.

[17] R. J. DiPerna, P. L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. **98** (1989) 511–547.

[18] R. M. Dudley, *Real Analysis and Probability*, Cambridge University Press, Cambridge, 2002.

[19] R. E. Edwards, *Functional Analysis, Theory and Applications*, Holt, Rinehart and Winston, 1965.

[20] J. Feng, D. Nualart, *Stochastic scalar conservation laws*, J. Funct. Anal. **255** (2) (2008) 313–373.

[21] F. Flandoli, *Dirichlet boundary value problem for stochastic parabolic equations: Compatibility relations and regularity of solutions*, Stoch. Stoch. Rep. **29** (3) (1990) 331–357.

[22] F. Flandoli, D. Gątarek, *Martingale and stationary solutions for stochastic Navier-Stokes equations*, Probab. Theory Related Fields **102** (3) (1995) 367–391.

[23] F. Flandoli, M. Gubinelli, E. Priola, *Well-posedness of the transport equation by stochastic perturbation*, Invent. Math. **180** (2010) 1–53.

[24] M. I. Freidlin, *On the factorization of nonnegative definite matrices*, Theory Probability Appl. **13** (1968) 354–356.

[25] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1964.

[26] G. Gagneux, M. Madaune-Tort, *Analyse mathématique de modèles non linéaires de l'ingénierie pétrolière*, Springer-Verlag, 1996.

[27] D. Gątarek, B. Gołdys, *On weak solutions of stochastic equations in Hilbert spaces*, Stochastics and Stochastics Reports **46** (1) (1994) 41–51.

[28] H. C. Grunau, W. von Wahl, *Regularity of weak solutions of semilinear parabolic systems of arbitrary order*, J. Anal. Math. **62** (1994) 307–322.

[29] I. Gyöngy, N. Krylov, *Existence of strong solutions for Itô's stochastic equations via approximations*, Probab. Theory Related Fields **105** (2) (1996) 143–158.

[30] I. Gyöngy, C. Rovira, *On L^p-solutions of semilinear stochastic partial differential equations*, Stochastic Process. Appl. **90** (2000) 83–108.

[31] M. Hofmanová, *A Bhatnagar-Gross-Krook approximation to stochastic scalar conservation laws*, to appear.

[32] M. Hofmanová, *Degenerate parabolic stochastic partial differential equations*, to appear.

[33] M. Hofmanová, *Strong solutions of semilinear stochastic partial differential equations*, Nonlinear Differ. Equ. Appl. **20** (3) (2013) 757–778.

[34] M. Hofmanová, J. Seidler, *On weak solutions of stochastic differential equations*, Stoch. Anal. Appl. **30** (1) (2012) 100–121.

[35] M. Hofmanová, J. Seidler, *On weak solutions of stochastic differential equations II.*, Stoch. Anal. Appl., to appear.

[36] H. Holden, N. H. Risebro, *Conservation laws with a random source*, Appl. Math. Optim. **36** (2) (1997) 229–241.

[37] N. Ikeda, S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd ed., North-Holland, Amsterdam, 1989.

[38] C. Imbert, J. Vovelle, *A kinetic formulation for multidimensional scalar conservation laws with boundary conditions and applications*, SIAM J. Math. Anal. **36** (1) (2004) 214–232.

[39] K. Itô, *On a stochastic integral equation*, Proceedings of the Japan Academy **22** (1–4) (1946) 32–35.

[40] K. Itô, *On stochastic differential equations*, Memoirs of the American Mathematical Society **4** (1951) 1–51.

[41] J. Jacod, A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, 2nd ed., Springer, Berlin 2003.

[42] I. Karatzas, S. Shreve, *Brownian Motion and Stochastic Calculus*, Springer, New York, 1988.

[43] R. Khasminskii, *Stochastic Stability of Differential Equations*, 2nd ed., Springer, Berlin, 2012.

[44] J. U. Kim, *On a stochastic scalar conservation law*, Indiana Univ. Math. J. **52** (1) (2003) 227–256.

[45] S. N. Kružkov, *First order quasilinear equations with several independent variables*, Mat. Sb. (N.S.) **81** (123) (1970) 228–255.

[46] N. V. Krylov, *A W_2^n-theory of the Dirichlet problem for SPDEs in general smooth domains*, Probab. Theory Related Fields **98** (3) (1994) 389–421.

[47] N. V. Krylov, B. L. Rozovskii, *On the Cauchy problem for linear stochastic partial differential equations*, Izv. Akad. Nauk. SSSR Ser. Mat. **41** (6) (1977) 1329-1347; English transl. Math. USSR Izv. **11** (1977).

[48] N. V. Krylov, B. L. Rozovskii, *Stochastic evolution equations*, Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. **14**, VINITI, Moscow, 1979, 71–146; English transl. J. Sov. Math. **16** (4) (1981) 1233–1277.

[49] H. Kunita, *Stochastic differential equations and stochastic flows of diffeomorphisms*, École d'été de Probabilités de Saint-Flour, XII-1982, Lecture Notes in Math., vol. 1097, 143–303, Springer, Berlin, 1984.

[50] H. Kunita, *Stochastic flows and stochastic differential equations*, Cambridge University Press, Cambridge, 1990.

[51] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs 23, Am. Math. Soc., Providence, R. I. (1968).

[52] E. H. Lieb, M. Loss, *Analysis*, 2nd ed., American Mathematical Society, Providence, 2001.

[53] G. M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[54] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod, Paris, 1969.

[55] P. L. Lions, B. Perthame, E. Tadmor, *Formulation cinétique des lois de conservation scalaires multidimensionnelles*, C.R. Acad. Sci. Paris (1991) 97–102, Série I.

[56] P. L. Lions, B. Perthame, E. Tadmor, *A kinetic formulation of multidimensional scalar conservation laws and related equations*, J. Amer. Math. Soc. **7** (1) (1994) 169–191.

[57] J. Málek, J. Nečas, M. Rokyta, M. Růžička, *Weak and Measure-valued Solutions to Evolutionary PDEs*, Chapman & Hall, London, Weinheim, New York, 1996.

[58] A. Nouri, A. Omrane, J. P. Vila, *Boundary conditions for scalar conservation laws from a kinetic point of view*, J. Statist. Phys. **94** (5-6) (1999) 779–804.

[59] A. Nouri, A. Omrane, J. P. Vila, *Erratum to "Boundary conditions for scalar conservation laws from a kinetic point of view"*, J. Statist. Phys. **115** (5-6) (2004) 1755–1756.

[60] M. Ondreját, *Stochastic nonlinear wave equations in local Sobolev spaces*, Electronic Journal of Probability **15** (33) (2010) 1041–1091.

[61] M. Ondreját, *Uniqueness for stochastic evolution equations in Banach spaces*, Dissertationes Mathematicae **426** (2004) 1–63.

[62] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equation*, Applied Mathematical Sciences, Vol. 44, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[63] B. Perthame, *Uniqueness and error estimates in first order quasilinear conservation laws via the kinetic entropy defect measure*, J. Math. Pures et Appl. **77** (1998) 1055–1064.

[64] B. Perthame, *Kinetic Formulation of Conservation Laws*, Oxford Lecture Ser. Math. Appl., vol. 21, Oxford University Press, Oxford, 2002.

[65] B. Perthame, E. Tadmor, *A kinetic equation with kinetic entropy functions for scalar conservation laws*, Comm. Math. Phys. **136** (3) (1991) 501–517.

[66] R. S. Philips, L. Sarason, *Elliptic-parabolic equations of the second order*, J. Math. Mach. **17** (1968) 891–917.

[67] P. E. Protter, *Stochastic Integration and Differential Equations*, Springer, 2004.

[68] T. Runst, W. Sickel, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*, de Gruyter Series in Nonlinear Analysis and Applications, vol. 3, Walter de Gruyter & Co., Berlin, 1996.

[69] S. S. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach, Yverdon, 1993.

[70] B. Saussereau, I. L. Stoica, Scalar conservation laws with fractional stochastic forcing: Existence, uniqueness and invariant measure, Stoch. Pr. Ap. **122** (2012) 1456–1486.

[71] A. V. Skorokhod, *On existence and uniqueness of solutions to stochastic diffusion equations*, (Russian) Sibirskiĭ Matematicheskiĭ Zhurnal **2** (1) (1961) 129–137.

[72] A. V. Skorokhod, *On stochastic differential equations*, (Russian) In Proceedings of the 6th All-Union Conference on Probability Theory and Mathematical Statistics, GIPNL Litovskoĭ SSR, Vil'nyus (1962) 159–168.

[73] D. W. Stroock, S.R.S. Varadhan, *Multidimensional Diffusion Processes*, Springer, Berlin, 1979.

[74] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland Publishing Company, Amsterdam, New York, Oxford, 1978.

[75] H. Triebel, *Theory of Function Spaces II*, Birkhäuser, Basel, 1992.

[76] G. Vallet, P. Wittbolt, *On a stochastic first order hyperbolic equation in a bounded domain*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **12** (4) (2009) 613–651.

[77] W. von Wahl, *Extension of a result of Ladyzhenskaya and Ural'ceva concerning second order parabolic equations of arbitrary order*, Ann. Pol. Math. **41** (1983) 63–72.

[78] Weinan E, K. Khanin, A. Mazel, Ya. Sinai, *Invariant measures for Burgers equation with stochastic forcing*, Annals of Mathematics **151** (2000) 877–960.

[79] A. Yagi, *Abstract Parabolic Evolution Equations and their Applications*, Springer-Verlag, Berlin, Heidelberg, 2010.

[80] X. Zhang, *Smooth solutions of non-linear stochastic partial differential equations driven by multiplicative noises*, Sci. China Math. **53** (2010) 2949–2972.

[81] X. Zhang, *Regularities for semilinear stochastic partial differential equations*, J. Funct. Anal. **249** (2007) 454–476.

Résumé

Dans cette thèse, on considère des problèmes issus de l'analyse d'EDP stochastiques paraboliques non-dégénérées et dégénérées, de lois de conservation hyperboliques stochastiques, et d'EDS avec coefficients continus.

Dans une première partie, on s'intéresse à des EDPS paraboliques dégénérées ; on adapte les notions de formulation et de solutions cinétiques, puis on établit l'existence, l'unicité ainsi que la dépendance continu en la condition initiale. Comme résultat préliminaire, on obtient la régularité des solutions dans le cas non-dégénéré, sous l'hypothèse que les coefficients sont suffisamment réguliers et ont des dérivées bornées.

Dans une deuxième partie, on considère des lois de conservation hyperboliques avec un forçage stochastique, et on étudie leur approximation au sens de Bhatnagar-Gross-Krook. En particulier, on décrit les lois de conservation comme limites hydrodynamiques du modèle BGK stochastique lorsque le paramètre d'échelle microscopique tend vers 0.

Dans une troisième partie, on donne une preuve nouvelle et élémentaire du théorème classique de Skorokhod, concernant l'existence de solutions faibles d'EDS à coefficients continus, sous une condition de type Lyapunov appropriée.

Abstract

In this thesis, we address several problems arising in the study of nondegenerate and degenerate parabolic SPDEs, stochastic hyperbolic conservation laws and SDEs with continues coefficients.

In the first part, we are interested in degenerate parabolic SPDEs, adapt the notion of kinetic formulation and kinetic solution and establish existence, uniqueness as well as continuous dependence on initial data. As a preliminary result we obtain regularity of solutions in the nondegenerate case under the hypothesis that all the coefficients are sufficiently smooth and have bounded derivatives.

In the second part, we consider hyperbolic conservation laws with stochastic forcing and study their approximations in the sense of Bhatnagar-Gross-Krook. In particular, we describe the conservation laws as a hydrodynamic limit of the stochastic BGK model as the microscopic scale vanishes.

In the last part, we provide a new and fairly elementary proof of Skorkhod's classical theorem on existence of weak solutions to SDEs with continuous coefficients satisfying a suitable Lyapunov condition.

N° d'ordre :

École normale supérieure de Cachan - Antenne de Bretagne

Campus de Ker Lann - Avenue Robert Schuman - 35170 BRUZ

Tél : +33(0)2 99 05 93 00 - Fax : +33(0)2 99 05 93 29

Charles University in Prague, Faculty of Mathematics and Physics

Ke Karlovu 3 - 121 16 Praha 2 - Czech Republic

tel: +420 22191 1289 - fax: +420 22191 1292